TRAJECTORIES TENDING TO A CRITICAL POINT IN 3-SPACE*

By RALPH E. GOMORY

1. Introduction

The object of this paper is to study the behavior of a solution to a system of ordinary differential equations, $dv/dt = F(v)$, v a real 3-vector, as this solution approaches a critical point P. The motion of the solution will be projected onto a unit sphere D around P, and the limit sets of the resulting projected motion will be studied. These limit sets, which characterize the asymptotic behavior of the solution as it tends to P, will turn out to be closely related to the solutions of a certain 2-dimensional system of differential equations defined on D. It will be proved, for example, that the limit set of the projected motion must contain some critical points of this 2-dimensional system, or else be a closed curve. Hence any solution tending to P must either return arbitrarily often arbitrarily near certain special directions, or else spiral asymptotically to a cone whose vertex is P.

The behavior of a trajectory tending to a critical point P in 2-space is known from [1]. The trajectory either tends to P asymptotically tangent to some line, or else spirals infinitely often around P. The behavior in 3-space will turn out to be considerably more varied. Certain of the theorems to be obtained for 3-space are valid also in higher dimensions.

2. The set $L(v)$

Consider the equation

$$(1) \qquad \frac{dv}{dt} = F(v) = \sum_{s=m}^{\infty} F_s(v),$$

where $F(v)$ is a real analytic vector function of the real 3-vector v, $v = (x, y, z)$, in some neighborhood of the origin, and $F_s(v)$ is the vector whose components are the terms of degree s in x, y, and z. We can assume that the critical point P is at the origin, so $F(0) = F_0(0) = 0$, with 0 the zero vector. Also let $m > 0$ be the first integer for which $F_s(v)$ is not identically zero.

Now let $v(t)$ be a solution of equation (1) tending to P as $t \to \infty$. The behavior of $v(t)$ will be studied by projecting its motion onto the unit sphere D around P. Let $v(t) = \sigma(t)u(t)$ where σ is the norm of v and u is a unit vector. Then, as $v(t) \to P$, $u(t)$ traces out a path on D which may of course be self-crossing. We will define the set $L(v)$ of limiting directions of approach of the solution $v(t)$ to be the positive limit set of the motion $u(t)$ on D. Thus a point u_0 on the unit sphere is a limiting direction of approach of the motion $v(t)$ if and only if there is a sequence t_n, $t_n \to \infty$, such that $u(t_n) \to u_0$.

If in the 2-dimensional case we form the one sphere D_1 of unit vectors, and define L on this, then the results mentioned in the introduction assert that L is either a single point of D_1, or all D_1. On D_1 then the possible structures for L are very limited. We will try to see what structures are possible for L on D.

3. E and E'

Let v' denote points in a second real 3-dimensional vector space E', and let u' denote points on its unit sphere D'. Let i be the mapping of E' onto the original space E which sends v' into the vector in E with the same components. Then the mapping $j: \sigma'u' \to \sigma u; \sigma' = 1 + \sigma, iu' = u$, sends D' and its exterior onto the original space E, with D' going into P. Outside D' however the map is one-one and analytic at every point, so there is an inverse, and this inverse carries a solution $v(t)$ of (1) into a motion $v'(t)$ in E'. Also as $v(t)$ tends to P, $v'(t)$ tends to D'.

As $v(t)$ satisfies (1), $v(t) = \sigma(t)u(t)$, and $\sigma(t) = (v(t), v(t))^{\frac{1}{2}}$ we have

(2)
$$\frac{d\sigma}{dt} = (u, F(v))$$

$$\frac{du}{dt} = \frac{1}{\sigma}\{F(v) - (u, F(v))u\},$$

with the round bracket denoting scalar product. Arranging in powers of σ gives

(3)
$$\frac{d\sigma}{dt} = \sum_{s=m}^{\infty} \sigma^s(u, F_s(u))$$

$$\frac{du}{dt} = \sum_{s=m}^{\infty} \sigma^{s-1}\{F_s(u) - (u, F_s(u))u\},$$

so for $\sigma' > 1$ the motion $v'(t) = \sigma'(t)u'(t)$ satisfies

(4)
$$\frac{d\sigma'}{dt} = \sum_{s=m}^{\infty} (\sigma' - 1)^s(u', F_s(u'))$$

$$\frac{du'}{dt} = \sum_{s=m}^{\infty} (\sigma - 1)^{s-1}\{F_s(u') - (u', F_s(u'))u'\}.$$

4. The set $L(v')$

Since as $t \to \infty$, $v'(t)$ tends to D', the ordinary positive limit set of $v'(t)$ consists of points on the unit sphere D'. This set will be called $L(v')$. A point u'_0 is in $L(v')$ if and only if there is a sequence $t_n \to \infty$ such that $v'(t_n) \to u'_0$. Clearly if there is such a sequence then $\sigma'(t_n) \to 1$, and $u'(t_n) \to u'_0$. Hence for $v(t)$, $\sigma(t_n) \to 0$ and $u(t_n) = iu'(t_n) \to iu'_0$. Therefore by the definition of $L(v)$, if u'_0 is in $L(v')$, iu'_0 is in $L(v)$ and conversely. Therefore $L(v) = iL(v')$ and $L(v')$ may be studied in place of $L(v)$.

It will be useful to reparametrize the motions $v'(t)$ in E' while preserving unchanged the limit sets $L(v')$. For this we need the following simple remark. Consider two analytic vector fields V_1 and V_2 defined throughout the same region R

and which differ on some open set A of R only by a continuous positive scalar factor. If p is a point of A the fields define two motions through p whose trajectories coincide as long as they remain in A. If both motions remain always in A they will both have the same positive limit set.

Now the equations

(5)
$$\frac{d\sigma'}{dt} = \sum_{s=m}^{\infty} (\sigma' - 1)^{s-m+1}(u', F_s(u'))u'$$

$$\frac{du'}{dt} = \sum_{s=m}^{\infty} (\sigma' - 1)^{s-m}\{F_s(u') - (u', F_s(u'))u'\},$$

are defined throughout E' and differ from equations (4) by the factor $(\sigma' - 1)^{m-1}$ which is positive for $\sigma' > 1$. Since $v(t)$ in E tends to P as $t \to \infty$, $v'(t)$ tends to D' and so lies always in the region $\sigma' > 1$. If we take a point p on $v'(t)$ and the trajectory $v''(t)$ of (5) through it, it too will tend to D' as t increases. However it can not reach D' at any finite time t_0, for on D' $d\sigma'/dt = 0$ and therefore a trajectory on D' at any time t_0 is always on D'. It follows that $v''(t)$ also remains in the region $\sigma' > 1$. Therefore the limit sets $L(v')$ and $L(v'')$ of $v'(t)$ and $v''(t)$ are the same. However, equations (5) have the important advantage of giving a non-trivial motion on D' itself.

5. Trajectories on D'

Trajectories of (5) starting on $\sigma' = 1$ remain on $\sigma' = 1$, their motion being given by

(6)
$$\frac{du'}{dt} = F_m(u') - (u', F_m(u'))u'.$$

The trajectories of this system on the sphere D' will give information about the structure of $L(v')$.

However in its present form (6) is a system of three equations in the three variables x', y', and z' with the restriction $x'^2 + y'^2 + z'^2 = 1$. In practice the form of the trajectories on and near D' can be investigated more easily by using local coordinates on D', so that the motion of (6) on D' is given by an ordinary 2-dimensional system and the motion of (5) near D' by an ordinary 3-dimensional one. For example consider only points (x', y', z') in E' with $z' > 0$. Let $\xi_1 = x'/z'$, $\xi_2 = y'/z'$, and $\zeta = z'(1 - (x'^2 + y'^2 + z'^2)^{-\frac{1}{2}})$. Then the coordinates ξ_1 and ξ_2 of a point p, $z' > 0$, specify a line through p which intersects D' in a point u'_0, and ζ is the projection on the z' axis of the distance from p to u'_0.

In terms of these coordinates the motion (5) becomes

(5a)
$$\frac{d\xi_1}{d\tau} = \sum_{s=m}^{\infty} \zeta^{s-m}\{X_s(\xi_1, \xi_2, 1) - \xi_1 Y_s(\xi_1, \xi_2, 1)\}$$

$$\frac{d\xi_2}{d\tau} = \sum_{s=m}^{\infty} \zeta^{s-m}\{Y_s(\xi_1, \xi_2, 1) - \xi_2 Y_s(\xi_1, \xi_2, 1)\}$$

$$\frac{d\zeta}{d\tau} = \sum_{s=m}^{\infty} \zeta^{s-m+1}Z_s(\xi_1, \xi_2, 1)$$

where $dt/d\tau = (1 + \xi_1^2 + \xi_2^2)^{\frac{1}{2}(m-1)}$, and $X_m(x, y, z)$, $Y_m(x, y, z)$, $Z_m(x, y, z)$ are the components of $F_m(v)$. Points (ξ_1, ξ_2, ζ) with $\zeta = 0$ are on D', so the motion of (6) on D' becomes the 2-dimensional system

(6a)
$$\frac{d\xi_1}{d\tau} = X_m(\xi_1, \xi_2, 1) - \xi_1 Z_m(\xi_1, \xi_2, 1)$$

$$\frac{d\xi_2}{d\tau} = Y_m(\xi_1, \xi_2, 1) - \xi_2 Z_m(\xi_1, \xi_2, 1)$$

with ξ_1 and ξ_2 local coordinates on D'.

6. General structure of $L(v')$

The positive limit set of a motion in a compact set is closed and connected. Also if a point q is in the limit set, and $T(q)$ is a trajectory through q, then all $T(q)$ is also in the limit set.

Thus if $v(t)$ passes through a point p and tends to P, and $v'(t)$ and $v''(t)$ are solutions to (4) and (5) passing through $j^{-1}p$, we have always $i^{-1}L(v) = L(v') = L(v'')$ and by the remarks above $L(v'')$ is a closed connected union of trajectories of (5) in D'. Hence

LEMMA 1. $L(v')$ *is a closed connected union of complete trajectories of the two-dimensional system* (6).

From this we have at once

THEOREM 1. *If $L(v')$ consists of a single point u_0', then u_0' must satisfy the equation $F_m(u_0') - (u_0', F_m(u_0'))u_0' = 0$.*

PROOF. As $L(v')$ contains u_0' it contains the complete trajectory through u_0' as well. As u_0' in all $L(v')$, the entire trajectory is just the one point u_0'. Therefore u_0' must be a singular point of (6). Hence the right hand side of (6) is zero at u_0'.

This theorem selects the possible asymptotic lines of approach to the critical point P. For if $v(t)$ tends to P asymptotically tangent to some line l, the set $L(v)$ will be a single point u_0, the intersection point of l and D, and this is only possible when $i^{-1}u_0 = u_0'$ satisfies the condition of Theorem 1. As no vector distribution on the sphere D' can be free of singular points, there are always some possible asymptotic lines of approach.

To obtain further information about $L(v')$ two lemmas will be useful.

Let $T^+(u')$ be the positive half trajectory of (6) through a point u' on D', and $T^-(u')$ the negative half trajectory. Then

LEMMA 2. *Let A and B be two disjoint open sets on the sphere D'. If $L(v') \cap A$ contains any complete trajectory of (6), and $L(v') \cap B$ is not empty, then there are points u_1' and u_2' on $L(v') \cap [D' - (A \cup B)]$ such that $T^+(u_1') \subset D' - B$ and $T^-(u_2') \subset D' - B$.*

PROOF. If C is any set on D', let $C(\varepsilon)$ be the set of points v', $v' = \sigma'u'$, where u' is in C and $1 \leqq \sigma' < 1 + \varepsilon$. Thus $A(\varepsilon)$ and $B(\varepsilon)$ are disjoint open sets in $D'(\varepsilon)$, and $D'(\varepsilon) - (A(\varepsilon) \cup B(\varepsilon))$ is closed in $D'(\varepsilon)$. From this it follows that any arc in $D'(\varepsilon)$ having points in $A(\varepsilon)$ and $B(\varepsilon)$ has points also in $D'(\varepsilon) - (A(\varepsilon) \cup B(\varepsilon))$.

Suppose now that the conditions for Lemma 2 are satisfied. Let p be the point of $L(v') = L(v'')$ whose complete trajectory $T(p)$ lies in A, and let $\bar{p}$ be a point of $L(v'')$ in B. By the definition of $L(v'')$ there are sequences $t_n \to \infty$, $\bar{t}_n \to \infty$, $v''(t_n) \to p$, $v''(\bar{t}_n) \to \bar{p}$. We may choose the sequences so that $\bar{t}_n < t_n < \bar{t}_{n+1} < t_{n+1}$. Since $v''(t) \to D'$, we may assume that for all $t > t_0$, $v''(t)$ is in $D'(\varepsilon)$.

For all n large enough $v''(t_n)$ will be in $A(\varepsilon)$ and $v''(\bar{t}_n)$ in $B(\varepsilon)$. Hence the closed arc of the trajectory of $v''(t)$ between $v''(\bar{t}_n)$ and $v''(t_n)$ must intersect $D'(\varepsilon) - (A(\varepsilon) \cup B(\varepsilon))$, and as $D'(\varepsilon) - (A(\varepsilon) \cup B(\varepsilon))$ is closed, there is in fact a first intersection point $e^-(n)$. Also, on the arc of trajectory between $v''(t_n)$ and $v''(\bar{t}_{n+1})$ there is a first intersection $e^+(n)$ with $D'(\varepsilon) - (A(\varepsilon) \cup B(\varepsilon))$. Thus we have a series of open arcs $E(n)$ in $A(\varepsilon)$ with end points $e^-(n)$, $e^+(n)$ in $D'(\varepsilon) - (A(\varepsilon) \cup B(\varepsilon))$. As $v''(t)$ tends to D', so do the arcs, and the $e^-(n)$, $e^+(n)$ will have points of accumulation on $D' - (A \cup B)$. By choosing a subsequence $\{e^-(n')\}$ we may assume that the $e^-(n')$ tend to E^- on D', and that the corresponding $e^+(n')$ tend to E^+. Clearly E^+ and E^- are in $L(v'')$. To prove Lemma 2 it is only necessary to show that $T^+(E^-)$ and $T^-(E^+)$ are in $D' - B$.

Let $T(t)$ be the transformation carrying each point into its position t seconds later under the motion given by (5). Suppose that for some $t_0 > 0$ $T(t_0)E^-$ is in B. Then we can find a neighborhood N of E^- in $D'(\varepsilon)$ such that $T(t_0)N \subset B(\varepsilon)$. In fact we may choose N so that even for the closure $\bar{N}$ we still have $T(t_0)\bar{N} \subset B(\varepsilon)$. Now N contains an infinity of end points $e^-(n')$, and under the transformation $T(t)$ each traces out a path which begins with its arc $E(n')$. Since under the transformation $T(t_0)$ each $e^-(n')$ goes into a point of $B(\varepsilon)$, $T(t)e^-(n')$ must already have cut $D'(\varepsilon) - (A(\varepsilon) \cup B(\varepsilon))$ for some t, $0 \leq t \leq t_0$, and $e^+(n')$ is the first possible intersection. Therefore the set U, $U = \bigcup_{t_0 \geq t \geq 0} T(t)\bar{N}$, contains

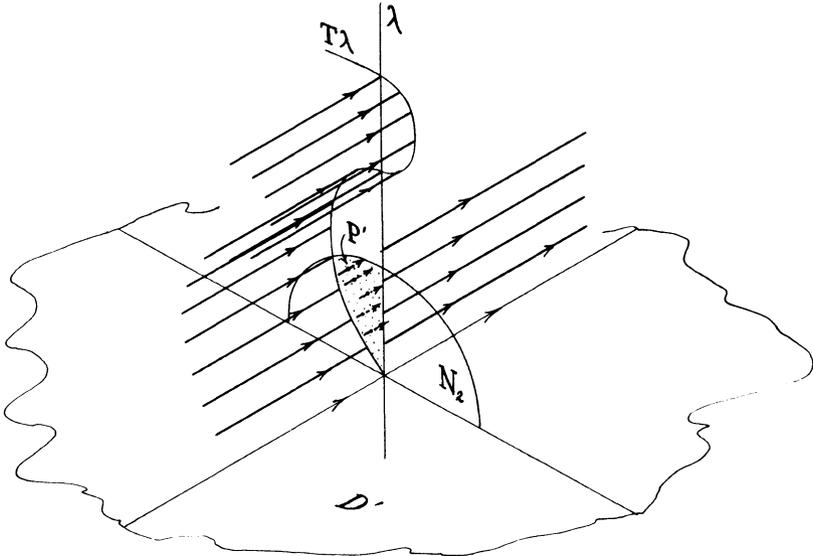
FIG. 1

an infinity of complete closed arcs $E(n')$. The points $v''(t_{n'})$ are on the $E(n')$ and tend to p. As U is closed, p is in U. But if p is in U then $T(t)p$ is in $T(t_0)\bar{N} \subset B(\varepsilon)$ for some $0 \leqq t \leqq t_0$. However the trajectory through p is in A for all t by hypothesis. This is a contradiction, so $T(t)E^-$ must be in $D - B$ for all $t > 0$.

If the transformation $T(-t)$, $t > 0$, is applied to E^+, the argument may be repeated to prove that $T(-t)E^+$ is in $D - B$ for all $t > 0$. This establishes Lemma 2.

LEMMA 3. *If C in D' is a closed curve solution of* (6), *then $L(v')$ does not have points in both the exterior and interior of C.*

PROOF. Let u_0' be a point of C, and let P be the plane normal to C at u_0'. As C is closed, the motion through u_0' returns and cuts P again at u_0'. It is well known that under these circumstances all the trajectories passing through any sufficiently small circular neighborhood N_1 of u_0' in P will cut P again, and that the transformation T which sends each point of N_1 into its trajectory's next intersection with P is an analytic map of N_1 into P.

Consider the arc λ in N_1 consisting of the points $\sigma u_0'$, $\sigma \geqq 1$. $T\lambda$ is again an analytic arc in P and has the same end point u_0'. Consider the intersections of λ with $T\lambda$, either, (a), there are only a finite number of intersections in some neighborhood of u_0', or, (b), the intersections accumulate at u_0'. In this second case analyticity implies that λ and $T\lambda$ must actually coincide throughout some neighborhood. In case (b) the trajectory arcs starting on λ and ending on $T\lambda$ form a band $\bar{B}$, and $\bar{B}$ will have a minimum height $\delta > 0$. Hence $D'(\delta) - \bar{B}$ consists of two disjoint sets I and E, open in $D'(\delta)$, with I containing the interior of C and E containing the exterior. It is clear that a trajectory starting on $\bar{B}$ remains on $\bar{B}$ while in $D'(\delta)$, so no trajectory in $D'(\delta)$ can cross from I to E.

Since the motion $v''(t)$ tends to D', it must lie in $D'(\delta)$ for $t > t_0$, and if $L(v'')$ has points in the interior of C, $v''(t)$ must be in I for some $t_1 > t_0$. Then $v''(t)$ can never be in E again, since it cannot cut $\bar{B}$ to cross from I to E. Therefore in case (b) $L(v'')$ can not have points in both the exterior and interior of C.

In case (a) choose a circular neighborhood N_2 in P so small that in N_2 λ and $T\lambda$ intersect only at u_0'. Then the band $\bar{B}$ formed by arcs of trajectories starting on λ in N_2 and ending on $T\lambda$ can be closed up by the portion P' of P lying between λ and $T\lambda$ in N_2. See Figure 1. So now $\bar{B} \cup P'$ divides some $D'(\delta)$ into sets E and I. If N_2 is taken small enough no vector through a point of P' is tangent to P, so trajectories cross P' only in one direction, say from E to I. As before, if $L(v'')$ has points in both I and E, then $v''(t)$ must cross $\bar{B} \cup P'$ from I to E, and this is impossible. Hence in this case too Lemma 3 holds.

These lemmas will now be applied to the various trajectories that are possible on D'.

7. Structure of $L(v')$ when D' has only a finite number of critical points

In this section it will be assumed that $F_m(u') - (u', F_m(u'))u'$ vanishes only at isolated points on D'. The various possible trajectories of (6) on D' will be considered one by one and related to the structure of $L(v')$.

7a. *Isolated critical points.* These critical points of the two-dimensional system (6) may be classified under the headings stable, unstable, stable-unstable, or centers. A critical point will be called stable if there are trajectories in D' tending to it with increasing t, and if no trajectory tends to it with decreasing t. A point is unstable if trajectories tend to it only with t decreasing. A point will be called stable-unstable if there are some trajectories tending to it with increasing t, and others with decreasing t. From the general classification of critical points of an analytic two-dimensional system, see [1], we have that any critical point which is neither stable nor unstable nor stable-unstable is a center. On the relation of isolated critical points to the set $L(v')$ there is the following theorem.

THEOREM 2. *If $L(v')$ contains an isolated critical point of (6) which is stable, unstable, or a center, then $L(v')$ is a single point.*

PROOF. Let p be a stable critical point in $L(v')$ and suppose that $L(v')$ contains any other point p'. Take a neighborhood $N(p)$ so small that p is the only critical point in the closure $\bar{N}(p)$, and so small that there are no closed solution curves in $\bar{N}(p)$. For a small enough $N(p)$, p' will be in $D' - \bar{N}(p)$, and since p is a complete trajectory, $N(p)$ and $D' - \bar{N}(p)$ are two sets satisfying the conditions for Lemma 2. Lemma 2 asserts that there are points u_1' and u_2' on $\bar{N}(p) - N(p)$ with $T^+(u_1')$ and $T^-(u_2')$ entirely in $\bar{N}(p)$. From the theory of limit sets, see [1], we know that the limit set of $T^-(u_2')$ lies in $\bar{N}(p)$ and is either a single critical point, a closed curve solution, or else consists of trajectories linking stable-unstable critical points. As there are no closed curves or stable-unstable points in $\bar{N}(p)$, the limit set of $T^-(u_2')$ would be the single critical point p, but $T^-(u_2')$ can not tend to p since p is stable. Therefore $L(v')$ contains no second point p' if p is stable. If p is unstable the argument may be repeated using $T^+(u_1')$ instead of $T^-(u_2')$.

If p is a center there are closed curve solutions C arbitrarily close to P which contain p in their interiors. Any other point p' of $L(v')$ would lie in the exterior region of some of these closed curves and this would contradict Lemma 3. This proves Theorem 2.

THEOREM 3. *If $L(v')$ contains an isolated critical point p of (6) which is stable-unstable, then either $L(v')$ is p, or there are positive and negative trajectories in $L(v')$ which tend to p.*

PROOF. Let the neighborhood $N(p)$ be formed as in the proof of Theorem 2. If p is not all $L(v')$ then, as before, there are trajectories $T^+(u_1')$, $T^-(u_2')$ which are in $L(v')$ and remain in $\bar{N}(p)$. If both trajectories tend to p the theorem holds. If one does not tend to p it tends to a closed curve or to a trajectory linking stable-unstable critical points. As we can take $\bar{N}(p)$ small enough to exclude closed curves, the second possibility is the only one, and the limit set consists of trajectories linking p to p. Since $L(v')$ is closed these linking trajectories are also in $L(v')$ and since they tend to p with both increasing and decreasing t the theorem holds in this case too.

7b. *Trajectories tending to a critical point.* From Theorem 2 we have at once

THEOREM 4. *If a trajectory T^+ (or T^-) of (6) tends to a stable or unstable isolated critical point p, then T^+ (or T^-) is not in $L(v')$.*

PROOF. If T is in $L(v')$ so is p, since $L(v')$ is closed. But then, by Theorem 2, p is all of $L(v')$.

7c. *Trajectories not tending to a critical point.* For trajectories that are not closed curve solutions we have the following.

THEOREM 5. *Let T^+ (or T^-) be a positive (or negative) half trajectory of (6) that does not tend to a critical point. If T^+ (or T^-) is not a closed curve solution it is not in $L(v')$.*

PROOF. If T^+ is not closed and does not tend to one critical point it must (a) spiral to a closed curve solution of (6), or (b) spiral to a graph whose vertices are critical points and whose sides are trajectories linking these critical points, with the whole graph forming the boundary of a two-cell E_2 containing T. See [2]. This last configuration will be called a generalized closed curve solution.

In case (a) if T^+ is in $L(v')$, so is the closed curve C, since $L(v')$ is closed. We may suppose that T^+ spirals to C from the interior region. Since T^+ spirals to C there are no closed curves in the interior region within some small distance δ of C. Let $B(\delta)$ be the set of points in the interior of C distant more than δ from C. If $\delta' < \delta$ is taken sufficiently small $B(\delta')$ will contain points of T^+. $B(\delta')$ is open and $D' - \bar{B}(\delta')$ is open and contains C, which is a complete trajectory in $L(v')$. Hence Lemma 2 applies and asserts that there is a trajectory $T^-(u_2')$ starting on $\bar{B}(\delta') - B(\delta')$ and remaining in $D' - B(\delta')$. This trajectory cannot cross C and hence remains in a bounded region free of singularities and closed curves other than C. Hence it must spiral to C. This is a contradiction as positive and negative trajectories cannot both tend to C from the interior. It follows that T^+ can not be in $L(v')$.

If instead of T^+ we have the negative half trajectory T^-, we need only replace $T^-(u_2')$ by $T^+(u_1')$ in the preceding proof.

In case (b) essentially the same proof applies. In place of the interior region of C, the two-cell E_2 is employed and the rest of the proof is unchanged. This completes the proof of Theorem 5.

The closed trajectories excluded in the preceding theorem are related to $L(v')$ in a very simple manner.

THEOREM 6. *Let C be a closed curve solution to (6). If C is in $L(v')$, then $C = L(v')$.*

PROOF. If $L(v') \neq C$ there are points of $L(v') - C$ arbitrarily near C since $L(v')$ is a connected set. We may assume there are points of $L(v') - C$ arbitrarily near C and in the interior region. If there are no closed curves arbitrarily near C in the interior region, the trajectories through points of $L(v') - C$ will be trajectories spiraling to C, see [1]. These trajectories however are in $L(v')$ as their starting points are in $L(v')$, and this contradicts Theorem 5. Again referring only to the interior of C, if there should be closed curves arbitrarily near C, we could select one containing any given point p' of $L(v') - C$ in its interior and containing C in its exterior. This closed curve would contain points of $L(v')$ in its in-

terior and its exterior and this contradicts Lemma 3. Thus in both cases $L(v') \neq C$ leads to a contradiction.

7d. *Possible sets $L(v')$.* On a sphere with isolated critical points the only possible trajectories are critical points, trajectories tending to a critical point, closed curve trajectories, trajectories spiraling to a closed curve solution, and trajectories spiraling to a generalized closed curve solution. See [2].

$L(v')$ is always a union of trajectories. By the preceding theorems trajectories tending either positively or negatively to critical points that are not stable-unstable can not be in $L(v')$. Also those that spiral positively or negatively to closed curve or generalized close curve solutions are ruled out. Using also Theorems 2, 3, and 6 we obtain Theorem 7.

THEOREM 7. *If D' has only isolated critical points, then $L(v')$ is a single critical point of* (6), *a single closed curve solution of* (6) *or a graph whose 1-cells are trajectories and whose vertices are stable-unstable critical points. In this last case each vertex is approached by a positive and a negative trajectory of the graph.*

Although the possible limit sets $L(v')$ resemble somewhat the possible limit sets of a motion on D', it is not true that a trajectory in $L(v')$ is necessarily in the limit set of some motion on D'. This is shown by the following example. Let the motion near the critical point be given by

$$\frac{dx}{dt} = (x^3 - x^2z + xy^2 - 2xyz + y^2z) + (-xz^3 + yz^3 - z^4)$$

$$\frac{dy}{dt} = (x^2y - 2xyz + y^3 - 2y^2z) + (-xz^3 - yz^3)$$

$$\frac{dz}{dt} = (x^2z + y^2z - 2yz^2) + (-z^4).$$

These equations give rise to a motion in E' near D'. We will consider only points in E' with $z' > 0$. For these points we may use the coordinates ξ_1, ξ_2, ζ described in Section 5. From (5a) we have that the motion near D' is given by

(5a')
$$\frac{d\xi_1}{d\tau} = -\xi_1^2 + \xi_2^2 + \zeta(\xi_2 - 1)$$

$$\frac{d\xi_2}{d\tau} = -2\xi_1\xi_2 + \zeta(-\xi_1)$$

$$\frac{d\zeta}{d\tau} = \zeta(\xi_1^2 + \xi_2^2 - 2\xi_2) + \zeta^2(-1).$$

On the sphere D', $\zeta = 0$ and we have

(6a')
$$\frac{d\xi_1}{d\tau} = -\xi_1^2 + \xi_2^2$$

$$\frac{d\xi_2}{d\tau} = -2\xi_1\xi_2,$$

so that $\xi_1 = \xi_2 = 0$ is a critical point. It can be verified directly that the family of circles $\xi_1^2 + (\xi_2 - a)^2 = a^2$ is a family of integral curves for this equation. Hence except on $\xi_2 = 0$ the trajectories move on these circles and all tend to the origin with increasing and decreasing time. See Figure 2. In particular the motion on the circle $a = 1$ is not in the limit set of any motion on D'. However it will appear that this trajectory and the critical point at the origin form the limit set of a trajectory $v''(t)$ tending to D'.
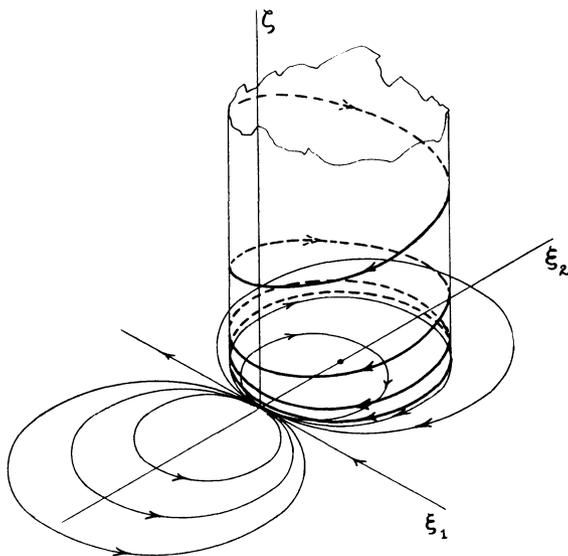

FIG. 2

Consider the equations (5a'). Let $v''(t)$ be a solution starting at any point $\zeta > 0$ of the cylinder $\xi_1^2 + (\xi_2 - 1)^2 = 1$. It can be verified directly that this cylinder is an integral surface of (5a') so a trajectory starting on it remains on it. On the cylinder $d\zeta/d\tau = -\zeta^2$ so $v''(t)$ tends toward $\zeta = 0$, and $\zeta = 0$ contains its limit set. This set is either the origin $\xi_1 = \xi_2 = \zeta = 0$ alone, or also includes the trajectory on the circle $\xi_1^2 + (\xi_2 - 1)^2 = 1$. If the origin alone is the limit set, $v''(t)$ eventually remains inside an ε-neighborhood of it. However near the origin and on the cylinder we have $-\xi_1^2 + \xi_2^2 < 0$ so $(d\xi_1/d\tau) < -\tfrac{1}{2}\zeta$. Hence

$$\mid \xi_1(\tau) - \xi_1(\tau_0) \mid > \frac{1}{2} \int_{\tau_0}^{\tau} \zeta(s)\, ds.$$

But $(d\zeta/d\tau) = -\zeta^2$, so $\zeta(\tau) = (\tau + \zeta^{-1}(\tau_0))^{-1}$ and therefore $\mid \xi_1(\tau) - \xi_1(\tau_0) \mid \to \infty$ as $\tau \to \infty$. So $v''(\tau)$ cannot remain in the ε-neighborhood, and its limit set must be both the origin and the trajectory on the circle $a = 1$.

It should be noted that Theorem 7 does not assert that there are only a finite number of 1-cells in the graph described in the Theorem. For if there is a closed

nodal region at some critical point on D', any finite or infinite union of trajectories from this closed nodal region will form an admissible graph.

8. Structure of $L(v')$ when D' has a curve of critical points

Suppose that $F_m(u') - (u', F_m(u'))u'$ vanishes for an infinity of points (x', y', z') on D', but not for all points on D'. For an infinity of these points one of the coordinates, say z', is not zero. Setting $x' = \xi_1 z'$ and $y' = \xi_2 z'$ and remembering that on D' where $z' \neq 0$, $(1 + \xi_1^2 + \xi_2^2)z'^2 = 1$, we have

$$(7) \qquad (1 + \xi_1^2 + \xi_2^2)^{\frac{1}{2}(m+2)}\{F_m(u') - (u', F_m(u'))u'\} = P(\xi_1, \xi_2)$$

where $P(\xi_1, \xi_2)$ is a vector whose components P_i are polynomials in ξ_1 and ξ_2 of degree $\leqq m + 2$. As the entire vector $P(\xi_1, \xi_2)$ vanishes for an infinity of values of ξ_1 and ξ_2, the P_i taken in pairs must have an infinity of zeros in common and so have common factors. Hence we may write $P(\xi_1, \xi_2) = P'(\xi_1, \xi_2)\phi'(\xi_1, \xi_2)$ where ϕ' is a polynomial in ξ_1 and ξ_2, and where the components of the vector P' have only a finite number of zeros in common. As $P(\xi_1, \xi_2) = P((x'/z'), (y'/z'))$ is of degree $\leqq m + 2$, multiplying both sides in (7) by $(z')^{m+2}$ gives on D'.

$$(8) \qquad F_m(u') - (u', F_m(u'))u' = P''(u')\phi''(u').$$

Here $\phi''(u')$ and the components of $P''(u')$ are polynomials in x', y', and z', and $P''(u')$ vanishes at only a finite number of points $z' \neq 0$ on D'. As an infinite of zeros on $z' = 0$ can then be dealt with in the same way, we may assume that $P''(u')$ has only a finite number of zeros on D'.

8a. *Possible trajectories on D'.* The trajectories of (6) on D' can now be studied by relating them to the trajectories of

$$(9) \qquad \frac{du'}{dt} = P''(u'), \quad \text{and (9a)}, \quad \frac{du'}{dt} = -P''(u').$$

Let u_0' be a point of D', with T^+ the positive trajectory of (6) through it. If $\phi''(u_0') = 0$, then u_0' is a critical point of (6) which is not necessarily isolated, and T^+ is a single point.

If $\phi''(u_0') > 0$, let T_1^+ be the positive trajectory of (9) through u_0', while if $\phi''(u_0') < 0$ let T_1^+ be the positive trajectory of (9a) through u_0'. Let A be the algebraic curve $\phi''(u') = 0$. There are then two possibilities, (1) T_1^+ intersects A, (2) T_1^+ does not intersect A.

In case (1) we may assume without loss of generality that $\phi''(u_0') > 0$. Then, up to the first intersection point p of T_1^+ and A, the trajectory T_1^+ lies in a region where (6) and (9) differ only by a positive scalar factor. Therefore the motion $u(t)$ of (6) through u_0' pursues the same path T_1^+ before the intersection point p, and tends to this critical point p as $t \to \infty$. In this case then T^+ is always a trajectory tending to a point on the curve A of critical points.

In case (2), as T_1^+ does not intersect A it lies always in the same component of $D' - A$, and therefore is always in an open region throughout which (6) and

(9) differ only by a positive scalar factor. Therefore the motion $u(t)$ of (6) has the same limit set as T_1^+. Hence T^+ has for its limit points a set which is; (a), a critical point of (9); (b), a closed curve solution of (9); (c), a generalized closed curve solution of (9). Since the limit set may contain points of A, in cases (b) and (c) it will not always consist of a closed curve or generalized closed curve solution to (6).

8b. *Structure of $L(v')$.* The possible trajectories on D' have been listed, now consider the possible limit sets $L(v')$. Theorems 2, 3, and 4 refer only to isolated critical points and trajectories tending to isolated critical points, and the proofs of these theorems are also unaffected by the existence of an infinity of zeros on D', so Theorems 2, 3, and 4 still hold. Theorem 5 holds too with a slight modification in the proof, for trajectories on D' not tending to a critical point tend to a closed curve solution or generalized closed curve solution of (9) or (9a) rather than of (6). However, as before, we can form the set $B(\delta')$ so that there are no closed curve solutions of (9) in $E_2 - B(\delta')$ where E_2 is the interior of the closed curve or generalized closed curve. Also, as T_1^+ does not cut A, and A has only a finite number of components, it is easily shown that there are no points of A in $E_2 - B(\delta')$ for δ' small enough. Now applying Lemma 2 we prove as before that if T^+ is in $L(v')$, then there is a trajectory $T^-(u_2')$ starting on $\bar{B}(\delta') - B(\delta')$ and remaining in $E_2 - B(\delta')$. As (9) and (6) differ in $E_2 - B(\delta')$ only by a positive scalar factor this implies the same behavior for a trajectory of (9) and this leads to the contradiction which originally proved Theorem 5. So Theorem 5 holds.

With similar changes we can prove a slightly modified Theorem 6.

THEOREM 6a. *If $L(v')$ contains a closed curve solution C of (9), then $L(v') = C$.*

Of course the closed curve solutions to (6) are closed curve solutions to (9).

Applying these theorems to the possible trajectories on D' we have Theorem 7a.

THEOREM 7a. *If D' has an infinity of critical points, but also some non-critical points, then $L(v')$ is a single critical point of (9), a closed curve solution of (9), or a union of points of A and trajectories which tend both positively and negatively either to points of A or to stable-unstable critical points.*

By applying Lemma 2 it is possible to obtain considerably more information about the sets occurring in the third case of Theorem 7a, however this will not be carried out here.

In both cases, with a finite or infinite number of critical points, we see from 7 and 7a that $L(v')$ either contains critical points, or is a closed curve C. Geometrically this means that a trajectory tending to a critical point either passes arbitrarily often arbitrarily close in direction to the unit vectors satisfying $F_m(u) - (u, F_m(u))u = 0$, or else must spiral in asymptotically to a cone. The only allowable cones being those whose intersection C with the unit sphere is a closed curve solution to (6). This remark also holds vacuously in the one case that still remains to be considered, and hence for all possible critical points.

9. Structure of $L(v')$ when every point of D' is critical

When $F_m(u') - (u', F_m(u'))u'$ vanishes on all D', the points $(u', F_m(u')) = 0$ on D' are identical with the points where the entire vector $F_m(u')$ vanishes, and as this vector was assumed not identically zero, $(u', F_m(u'))$ is not zero for all points of D'.

Consider then the equations

$$\frac{d\sigma'}{dt} = \sum_{s=m}^{\infty} (\sigma' - 1)^{s-m} (u', F_s(u'))$$

(10)

$$\frac{du'}{dt} = \sum_{s=m+1}^{\infty} (\sigma' - 1)^{s-(m+1)} \{F_s(u') - (u', F_s(u'))u'\}$$

which differ from equations (5) by a factor $(\sigma' - 1)$. On D' (10) becomes

$$(11) \qquad \frac{d\sigma'}{dt} = (u', F_m(u')), \qquad \frac{du'}{dt} = F_{m+1}(u') - (u', F_{m+1}(u'))u'.$$

Now let $v''(t)$ be a motion of (5) tending to D', and let $v'''(t)$ be a motion of (10) through a point of the trajectory of $v''(t)$. $v'''(t)$ traces through the trajectory of $v''(t)$ and either tends to D' as $t \to \infty$, or reaches D' for some $t = t_0$.

If $v'''(t)$ tends to D' as $t \to \infty$ it can contain in its limit set only points of D' and hence only points whose entire trajectories are in D'. Call the set of these trajectories S. Since in this case $v'''(t)$ and $v''(t)$ remain in a region where the vector fields of (5) and (10) differ only by a positive scalar, their limit sets are the same, and therefore $L(v') \subset S$. Also S is clearly a subset of the set of points of D' satisfying $d\sigma'/dt = 0$. Hence for every point of S, $F_m(u') = 0$.

In the other case let t_0 be the first t for which $v'''(t_0)$ is on D'. Then clearly $v''(t)$ tends to this point as $t \to \infty$, so $L(v') = (v'''(t_0))$. Therefore Theorem 8 holds.

THEOREM 8. *If $F_m(u') - (u', F_m(u'))u'$ is always zero on D', then $L(v')$ is either a single point or a connected set on which all components of F_m vanish.*

This implies that unless the components of F_m have some common factor, $L(v')$ is always a point.

9a. *Trajectories tending to P.* If $F_m(u') - (u', F_m(u'))u'$ is always zero it is easy to show that there actually are trajectories tending to the original critical point P.

Clearly a trajectory of (10) cuts D' at every point where $d\sigma'/dt \neq 0$. The portion of such a trajectory outside D' is a path of a motion of (5) which traverses the trajectory in one direction or the other and hence tends to D' either as $t \to \infty$ or as $t \to -\infty$. Therefore we have Theorem 9.

THEOREM 9. *If $F_m(u') - (u', F_m(u'))u' = 0$ for all unit vectors u', then there is a positive or negative trajectory tending to the critical point asymptotically tangent to every line whose unit vector satisfies the inequality $F_m(u') \neq 0$.*

10. Extension to higher dimensions

Theorems 1, 8, and 9 still hold if the v in equation (1) is an n-vector. The proofs are unchanged.

PRINCETON UNIVERSITY

REFERENCES

[1] BENDIXSON, I. *Sur les courbes définies par des équations différentielles.* Acta Math., Vol. 24, 1901, pages 1–88.
[2] LEFSCHETZ, S. Lectures on differential equations. Princeton University Press, 1946.