As a result linear programmers rather than electrical network theorists will probably find this book of greatest interest; for example, the tree concept, which is of basic importance in network applications, is first introduced in the last chapter, where multiterminal maximal flows are treated. Finally, it should be mentioned that an English translation [6] of Berge's book has been made.

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VII. COMMUNICATION NETWORKS

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The networks of this discussion are simplified models of a real communication net. They consist of nodes N_i, representing communication centers, and arcs B_{ij} directly linking the ith and jth centers; these arcs represent communication links. Associated with each arc is its "capacity" b_{ij} which measures its ability to transmit messages. Within this framework, following Ford and Fulkerson, we introduce the concept of "flow" which in this model represents message flow. A $p - q$ flow is a set of non-negative numbers x_{ij} individually representing the message flow or bits per second through B_{ij}, such that $x_{ij} \leq b_{ij}$, and

$$\sum_i x_{ij} - \sum_k x_{jk} - 0 \qquad j \neq p, q. \tag{1}$$

A $p - q$ flow then corresponds to the sending of messages from p to q using the intermediate nodes or stations as relay points; the conditions (1) assure that all messages entering an intermediate node are sent on again. The flow value F_{pq} measures the amount of messages emanating

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from p (or equivalently received at q) and so is given by

$$F_{pq} = \sum_k x_{pk}. \tag{2}$$

For a given network, the maximal value that F_{pq} can attain measures the ability of the whole network to transmit from p to q and this maximal flow value will be denoted by F_{pq}. Due to the work of Ford and Fulkerson [4], F_{pq} is readily calculable. Among the problems that arise naturally in connection with this model are problems of analysis; *i.e.*, given a network, is it capable of handling what flows; and problems of synthesis; *i.e.*, given certain flow requirements, how does one construct a network that will handle those flows. The problems of synthesis usually involve a cost minimization. A cost c_{ij} is assessed for each unit of capacity b_{ij} used and the full synthesis problem is to construct a network capable of transmitting certain flows and minimal in cost.

Most of these problems are naturally expressible as large linear programming problems; see, for example, Juncosa-Kalaba [12]. However, the direct expression of communication network problems in this form usually involves an enormous matrix, with one column for every path in the network or something else of the same great magnitude. Thus, little progress is made until either this formulation is replaced by a more special one or else the special structure of the matrices can be exploited to reduce the magnitude of the calculation involved in the linear programming formulation. In our discussion, then we will confine ourselves to the cases where special methods have been used or where the linear programming problem has been greatly reduced.

By far the greatest progress has been made in the area of time-shared networks. In a time-shared network the entire network is first used to send messages from p to q, then the whole network used to send from p' to q', and so on through all pairs of nodes. Its capabilities are then measured by the F_{pq} values for all p and q. (This F_{pq} forms a matrix which is sometimes known as the terminal capacity matrix.) The main results are as follows.

Symmetric Time-Shared Networks

In the symmetric network we assume $b_{ij} = b_{ji}$; *i.e.*, a link transmits equally well in either direction. Before taking up the problem of analysis of the symmetric time-shared network, let us consider a problem called the problem of realizability which for this class of networks sheds a great deal of light on the whole situation. The realizability question, first posed by Mayeda [14], is this: when is a matrix of numbers R_{pq} the terminal capacity matrix of some symmetric network? Certainly we must have $F_{pq} = F_{qp}$, because of the symmetry, but much more is needed. Mayeda [14] gave a condition for an R_{pq} matrix to be realizable. The condition was that the matrix of the R_{pq}, after rearrangement of rows and columns, had to be partionable into certain blocks, which were in turn partionable, etc. If the partition could be effected, the R_{pq} matrix was realizable, otherwise not, so that condition was necessary and sufficient; and also, once the matrix was partitioned a procedure was given for actually constructing a network with $F_{pq} = R_{pq}$. Thus, the network was even synthesized, though not at minimum cost. However, no procedure for actually finding the rearrangement of rows and columns was given. Next, Gomory and Hu [7] showed that a necessary and sufficient condition for realizability was the triangle inequality

$$R_{pq} \geq \min (R_{pk}, R_{kq}) \quad \text{all} \quad p, q, k.$$

This condition has the advantage that it can be easily checked since the procedure of checking can be reduced to the maximal spanning tree computation of Kruskal [13] and Prim [16]. Also the maximal spanning tree obtained from the computation gives a synthesis, again not at minimal cost.

The analysis problem for these networks is to find the $n(n-1)/2$ distinct F_{pq} values. Of course this can be done by solving $n(n-1)/2$ maximal flow problems by the methods of Ford and Fulkerson, each computation giving one of the F_{pq}. However, the realizability results suggest that one can do better, and Gomory and Hu [7] describe a method in which all $n(n-1)/2$ F_{pq} are obtained after only $n-1$ maximal flow computations.

The minimum cost synthesis problem for these networks can be divided into two parts. In the first case, the unit cost c_{ij} of capacity in the arc B_{ij} is taken to be the same in each arc (*i.e.*, $c_{ij} = 1$) and the problem is to find a network whose flows equal or exceed certain requirements: *i.e.*, $F_{pq} \geq R_{pq}$ and for which the total cost, which in this case is $\sum_{i,j} b_{ij}$ is minimal.

This problem was first solved by Chien [1] who gave a direct construction for a minimal cost network, if the requirements can be put into Mayeda's partitioned matrix form. Another minimum synthesis procedure which also depends on the ability to put the requirement matrix into Mayeda's partitioned form was given by Wing and Chien [20]. Later, Gomory and Hu [7] gave more general procedures which do not depend on Mayeda's partitioned form, and are easily constructed. None of these constructions involved linear programming but were direct special methods. An interesting fact that emerged is that among the many minimal cost networks satisfying the requirements there is a dominant one; that is, there is one minimal cost network whose maximal flow values F'_{pq} satisfy $F'_{pq} \geq F_{pq}$ for all pq where the F_{pq} are obtained from any other minimum cost network satisfying the flow requirements.

Most of the pleasant features of the problem disappear when the unit costs c_{ij} are no longer equal to 1, and the problem is to minimize $c_{ij}b_{ij}$. The problem was formulated as a very large linear programming problem by Wing and Chien [20]. An effective procedure exploiting the special structure of the linear programming problem was given by Gomory and Hu [8].

Even though the symmetric time-shared network is the one on which the greatest progress has been made, many simple-sounding unsolved problems remain. For example, there is as yet no procedure for finding, even when all c_{ij} are 1, the minimal cost network with the smallest number of arcs, or of dealing with problems where the unit costs of additional capacity decrease as the capacity of the individual arc increases. Yet, this situation is often the case in practical situations.

When we turn to the nonsymmetric time-shared case, we are already in difficulty. In the area of realizability it is known that the condition [7] although necessary is not sufficient for a network of more than three nodes (Tang and Chien [18]). This difficulty is reflected in the lack of results in the area of analysis and the very scattered and special nature of the results in synthesis. Although the linear programming approach of [8] can be pushed through, the number of variables is doubled due to the lack of symmetry. Also, the use of a dominant subset of requirements which imply all the others and which was used effectively in the symmetric case is not as effective here since the dominant subset does not consist of a fixed small number of elements [10], [15].

So far, we have discussed flows involving only one type of flow at a time, our networks at any instant handled only messages from p to q etc. In general, a communication network must handle messages from p to q, p' to q', etc., simultaneously. This leads us into the difficult area of multicommodity flows.

Multicommodity Flows

Let us consider first the steady-state case where there are requirements R_{pq} for flows that must take place simultaneously in the network. Here, least cost synthesis presents no difficulties. It is easily seen that the cheapest network is obtained by connecting each pair of nodes N_p, N_q by the cheapest path, each link having capacity R_{pq}, then adding up all paths. This can be done with the help of the usual shortest path methods [2].

When we come to analysis, or determining what sets of multicommodity flows are possible in a given network, the situation is more complicated. Ford and Fulkerson have given a special linear programming procedure for determining a maximal total flow in a network [5] and a great deal of effort has been expended on attempts to generalize to the multicommodity case the max-flow min-cut theorem of Ford and Fulkerson which gives a description of the possible flows in the one-terminal-pair one-commodity case. Hakimi [9] gave a generalization of the max-flow min-cut theorem in [3]. Unfortunately, counterexamples to his extension now exist, and it is believed to be true only in the case of two commodity flows. Hu [11] has given a special labelling procedure for obtaining maximal two-commodity flows in a paper that contains a proof of Hakimi's conjecture for the two-commodity case.

Time Varying Multi-Commodity Flows

Even more complicated is the situation when the simultaneous requirements R_{pq} vary over time. We are then confronted with the need to analyse or synthesize networks that are to provide flows R_{pq}^{t} at time t and flows $R_{pq}^{t'}$ at time t', etc. Little work has been done along these lines.

Tang [17] takes up the problem in which all unit costs $c_{ij} = 1$ and confines the network to trees. Although in its present published form the presentation in [17] depends on Hakimi's incorrect results and is therefore also incorrect, it can easily be justified by direct methods [19].

It is clear from this summary that a great deal of work still needs to be done in this area. One further aspect of the theory which is quite undeveloped is the treatment of reliability; *i.e.*, of networks in which some links sometimes fail to transmit. A typical problem in this area might be to find the probability of getting a message from p to q in a network in which the ijth arc fails with probability p_{ij}. This could be done by looking at all $p - q$ paths and computing the probability of an unbroken path. An approach which would seem to be some improvement on this is given by Fu and Yau [6]. In the reliability area there is not yet full agreement on either the model or the problems.

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VIII. SWITCHING FUNCTIONS

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INTRODUCTION

IN RECENT YEARS the activity in the general area of switching theory or logical design theory has been expanding quite rapidly. In this report we will first attempt to summarize some of the general aspects of this activity such as conferences, publications, and organizations. We will then outline what we consider to be the major technical areas in which the activity in combinational circuits or switching functions has been concentrated.

MAJOR ACTIVITIES

The single event of greatest significance for research in this field which occurred during the past three years was the formation within the AIEE of a subcommittee on Logic and Switching Circuit Theory. This subcommittee was formed under the AIEE Computing Devices Committee during the Fall of 1959. Its major activity has been the sponsoring of an annual symposium on switching circuit theory and logical design. The first of these symposia was held in October of 1960 in connection with the AIEE Fall General Meeting and National Electronics Conference in Chicago, Ill. This meeting was followed by two subsequent meetings held in October of 1961 and 1962. While there were no proceedings published for the first annual symposium, many of the papers presented are available in the publication containing the proceedings of the Second Annual Symposium [A]. This publication contains all of the papers from the Second Annual Symposium. The quantities in which this publication was made available were quite small and it is not available at the present time. A proceedings was also prepared for the Third Annual Symposium [B].

Early in 1962 this AIEE subcommittee was recognized as a Technical Activities Committee of the Professional Group on Electronic Computers of the IRE. This step was taken in anticipation of the merger of the AIEE and IRE, and also to avoid unnecessary duplication of effort. At approximately the same time a decision was also made to divide the IRE Transactions on Electronic Computers into sections corresponding to the main technical subdivisions of the field, one of these being Logic and Switching Theory.

In addition to the Annual Symposium sponsored by the AIEE subcommittee several meetings there were also held, either exclusively in the fields of switching theory or in very closely related areas. Of major significance are the following conferences:

[C] Symposium on Mathematical Theory of Automata, sponsored by the Polytechnic Institute of Brooklyn and held on April 24–26, 1962. Many papers at this conference were concerned with novel applications of various branches of abstract algebra to problems in automata theory.

[D] The Symposium on the Application of Switching Theory and Space Technology. This conference was held in February, 1962 in Sunnyvale, Calif., and was sponsored jointly by the United States Air Force and the Lockheed Missile and Space Company. The subject areas covered at this conference were quite general. Of particular note was the large number of scientists from outside the United States who presented papers at the meeting.

[E] A Symposium on Redundancy Techniques for Computing Systems. This Symposium was held in February, 1962, and was sponsored by the Information Systems Branch of the Office of Naval Research and Electronics Division of the Westinghouse Electric Corporation. This conference is particularly notable because it presents a picture of the state of the art techniques for improving reliability in logical networks as of the time it was held.

[F] IFIP Congress-62 held in August of 1962 in Munich, Germany. While this was a general conference in the Information Processing field there were both a paper session and a symposium session held on switching theory.

[G] International Symposium on Relay Systems and Finite Automata Theory sponsored by the IFAC Technical Committee on Theory and held in Moscow, U. S. S. R., in September, 1962. This meeting is of particular importance because it is the first meeting devoted exclusively to Automata Theory to be held in the USSR. There were approximately eighty papers presented at this conference by scientists from the USSR, the USA, Italy, Romania, Canada, Poland, Switzerland, France, Hungary, Sweden, and Austria.

SUBJECT AREAS

One of the major areas of activity continues to be that of minimization and simplification of combinational networks. Much of the work in this area is now directed

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