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# A STUDY OF TRAJECTORIES WHICH TEND TO A LIMIT CYCLE IN THREE-SPACE

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## 1. Introduction

The object of this paper is to study the behavior of a solution curve of an ordinary differential equation as this solution curve approaches a limit cycle of the differential equation in Euclidian 3-space. In the entire paper we shall denote the limit cycle by the symbol $C$ and the solution curve which tends to $C$ by the symbol $S$. The same problem if studied in 2-space has a very simple solution. A solution curve which approaches a limit cycle in the plane spirals towards this limit cycle from one of the two sides of the limit cycle. The manner in which a trajectory can approach a limit cycle in 3-space was studied by Birkhoff [1]. But in his classical paper on transformations of surfaces Birkhoff assumes conservation of energy. No such assumption will be made here.

The methods of this paper are similar to the methods used by one of us [2] in investigating singular points in 3-space. We shall use some of the results obtained there.

In Section 2 the exact problem is stated and certain transformations of coordinates carried out which facilitate the analysis. In Section 3 two tori, $T_1$ and $T_2$, are constructed. $T_1$ is essentially the torus of radius $\varepsilon$ with the limit cycle as center line. The other torus $T_2$ is produced roughly by expanding $C$ in another 3-space. If $i$ is the identity map from the second 3-space onto the first one it is then shown that a certain set on $T_1$ which is related to the limiting behavior of $S$ is just the image under $i$ of a set on $T_2$ about which it is fairly simple to get information. Section 4 is devoted to the study of limit sets on $T_2$. Finally, in Section 5 the main theorem is proved which shows how a solution curve can behave if it approaches a limit cycle.

## 2. Statement of the Problem

In the following we shall study the differential equation

$$dX/dt = F(X) \tag{2.1}$$

where $X$ and $F$ are 3-vectors and it is assumed that the components of $F$ are analytic functions in the components of $X$.

We assume that (2.1) admits a limit cycle, $C$, and a solution, $S$, which tends to the limit cycle in the following sense: For any fixed choice of the parameter $t$ and any fixed $\varepsilon > 0$ there exists $T$ such that for $t > T$ $S(t)$ is within $\varepsilon$ of $C$.

Near $C$ it will be useful to adopt coordinates based on the limit-cycle itself. In this local system one coordinate, $\theta$, runs along the limit cycle, while two others, $y_1$ and $y_2$, give the position relative to mutually perpendicular axes in the plane normal to the limit-cycle at the point specified by $\theta$. That these coordinates exist and are in fact analytic seems well known. See for instance a forthcoming book by S. Lefschetz [3] or a paper by Diliberto and Hufford [4]. Specifically then we take the following lemma as known.

LEMMA 2.1. *There exist coordinates $y_1$, $y_2$, and $\theta$ valid in some neighborhood of $C$ such that*

(a) *$y_1$, $y_2$, and $\theta$ are analytic functions of the $x_i$;*

(b) $(y_1, y_2, \theta)$ and $(y_1, y_2, \theta + 1)$ correspond to the same point;

(c) *if $p \in C$ then $y_i(p) = 0$, $dy_i/dt\ (p) = 0$, and $d\theta/dt\ (p) = 1$;*

(d) *if* (2.1) *transforms into the system*

$$\begin{aligned} dy_i/dt &= Y_i \qquad i = 1,2 \\ d\theta/dt &= Y_3 \end{aligned} \tag{2.2}$$

*then the right hand side of* (2.2) *consists of power series in $y_1$ and $y_2$ with coefficients which are periodic functions of $\theta$.*

Remark: (c) and (d) together imply that (2.2) is of the following form:

$$\begin{cases} dy_1/dt = \sum_{k=1}^{\infty} Y_1^k \\ dy_2/dt = \sum_{k=1}^{\infty} Y_2^k \\ d\theta/dt = 1 + \sum_{i=1}^{\infty} D_i \end{cases} \tag{2.3}$$

where $Y_i^k$ is a polynomial in $y_1$ and $y_2$ (combined powers of $y_1$ and $y_2$ equal to $k$) with coefficients which are periodic functions of $\theta$, and where $D_i$ is a polynomial in $y_1$ and $y_2$ (combined powers of $y_1$ and $y_2$ equal to $k$) with coefficients which are periodic functions of $\theta$. We next perform one more change in coordinates. Let

$$\sigma = (y_1^2 + y_2^2)^{-\frac{1}{2}}.$$

Let $u$ be the unit vector $(y_1/\sigma, y_2/\sigma)$ and $u_1$ and $u_2$ the components of $u$. We obtain the following:

$$\begin{aligned} d\sigma/dt &= 1/\sigma\ \{y_1[Y_1^N + Y_1^{N+1} + \cdots] + y_2[Y_2^N + Y_2^{N+1} + \cdots]\} \\ &= 1/\sigma\ \{\sigma u_1[\sigma^N Y_1^N(u_1, u_2, \theta) + \sigma^{N+1} Y_1^{N+1}(u_1, u_2, \theta) + \cdots] \\ &\qquad + \sigma u_2[\sigma^N Y_2^N(u_1, u_2, \theta) + \sigma^{N+1} Y_2^{N+1}(u_1, u_2, \theta) + \cdots]\} \\ &= \sum_{S=N}^{\infty} \sigma^S A_S(u_1, u_2, \theta) \end{aligned}$$

$$\begin{aligned} \frac{du}{dt} &= \left\{\frac{\sigma(dy_1/dt) - y_1(d\sigma/dt)}{\sigma^2}, \frac{\sigma(dy_2/dt) - y_2(d\sigma/dt)}{\sigma^2}\right\} \\ &= 1/\sigma\{dy_1/dt, dy_2/dt\} - \frac{d\sigma/dt}{\sigma^2}(y_1, y_2) \\ &= 1/\sigma\{\sigma^N Y_1^N(u_1, u_2, \theta) + \sigma^{N+1} Y_2^N(u_1, u_2, \theta) + \cdots, \sigma^N Y_2^N(u_1, u_2, \theta) \\ &\qquad + \sigma^{N+1} Y_2^{N+1}(u_1, u_2, \theta) + \cdots\} - \frac{\sum_{S=N}^{\infty} \sigma^S A_S(u_1, u_2, \theta)}{\sigma^2}\{\sigma u_1, \sigma u_2\} \\ &= \sum_{\sigma=N-1}^{\infty} \sigma^S B_S(u_1, u_2, \theta) \text{ where } B_S \text{ is a two-vector.} \end{aligned}$$

And finally the following system is obtained

$$(2.4)\qquad \begin{cases} d\theta/dt = 1 + \sum_{i=1}^{\infty} \sigma^i D_i(u_1, u_2, \theta) \\ d\sigma/dt = \sum_{S=1}^{\infty} \sigma^S A_S(u_1, u_2, \theta) \\ du/dt = \sum_{S=0}^{\infty} \sigma^S B_S(u_1, u_2, \theta) \end{cases}$$

where the $A_S$, $B_S$, and $D_i$ are periodic in $\theta$ and where any of the $A_S$, $B_S$, and $D_i$ may be identically zero.

## 3. The Tori $T_1$ and $T_2$

In introducing the coordinates $y_1$, $y_2$, and $\sigma$ in Section 2 we could have also noted that for sufficiently small $\varepsilon$ the set, $y_1^2 + y_2^2 = \varepsilon$ ($\sigma = \varepsilon$), is homeomorphic to a torus with $C$ as center line. We shall denote this set with $\varepsilon$ fixed henceforth by $T_1$. $S$ will be inside $T_1$ for $t \geqq t_0$ for some $t_0$. We project the part of $S$ corresponding to $t \geqq t_0$ onto $T_1$ and denote it by $P(S)$. Under $P$ the point $(y_1, y_2, \theta)$ maps onto the point $(\varepsilon y_1/\sigma, \varepsilon y_2/\sigma, \theta)$. A point $q \varepsilon T_1$ is said to be in the limit set of $P(S)$ if there exists a sequence $\{t_n \mid t_n \to \infty\}$ and a sequence $\{\varepsilon_n \mid \varepsilon_n \to 0\}$ such that $P[S(t_n)]$ is within $\varepsilon_n$ of $q$. We shall denote the limit set of $P(S)$ by $L$. This limit set is of interest because it characterizes the limiting motions of the trajectory $S$ near $C$. A direct study of $L$, however, is difficult since $P(S)$ may cross itself on $T_1$. For this reason the following artifice is introduced:

Let $E_3$ be the Euclidean 3-space containing $C$ and $T_1$. We now consider another Euclidean 3-space, $E_3'$, and two maps $i$ and $f$ from $E_3'$ into $E_3$. $i$ is just the identity map of $E_3'$ onto $E_3$. $i^{-1}$ maps $C$ and $T_1$ onto a closed curve and a torus in $E_3'$ respectively which we shall denote by $C'$ and $T_2$ (See Figure 1). In a neighborhood of $C'$ we introduce coordinates $\theta'$, $\sigma'$, and $u'$ which are defined as follows:

If $q$ is sufficiently near $C$ in $E_3'$

$$\theta'(q) = \theta(iq)$$
$$\sigma'(q) = \sigma(iq)$$
$$u'(q) = u(iq).$$

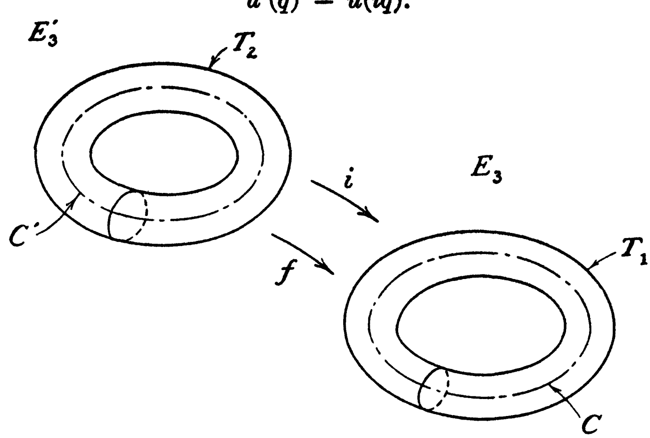


FIG. 1

The second map, $f$, is defined only in the neighborhood of $C'$ which is covered by the $\theta' - \sigma' - u'$ coordinate system and for which $\sigma \geqq \varepsilon$. If $q$ belongs to this neighborhood $f(q)$ is defined in the following way:

$$\theta[f(q)] = \theta'(q)$$
$$u[f(q)] = u'(q)$$
$$\sigma[f(q')] = \sigma'(q) - \varepsilon.$$

We note that $f$ maps $T_2$ onto $C$ and an open neighborhood of $T_2$ analytically and homeomorphically onto an open neighborhood of $C$. Intuitively this corresponds to expanding $C$ into the torus $T_2$. $f^{-1}$ when applied to (2.4) in an open neighborhood of $C$ excluding $C$ itself induces a differential equation in an open neighborhood of $T_2$ excluding $T_2$ itself. We obtain

$$(3.1) \quad \begin{cases} d\theta'/dt = 1 + \sum_{i=1}^{\infty} (\sigma' - \varepsilon)^i D_i(u_1', u_2', \theta') \\ d\sigma'/d = \sum_{S=1}^{\infty} (\sigma' - \varepsilon)^S A_S(u_1', u_2', \theta') \\ du'/dt = \sum_{S=1}^{\infty} (\sigma' - \varepsilon)^S B_S(u_1', u_2', \theta'). \end{cases}$$

It is clear that $f^{-1}(S)$ will be a trajectory of (3.1) which tends to $T_2$. We shall denote this trajectory by $S'$ and its limit set on $T_2$ by $L'$. Specifically a point $q$ of $T_2$ belongs to $L'$ if there exist sequences $\{t_n \mid t_n \to \infty\}$ and $\{\varepsilon_n \mid \varepsilon \to 0\}$ such that $S'(t_n)$ is within $\varepsilon_n$ of $q$.

LEMMA 3.1. *$q \varepsilon L' \Leftrightarrow iq \varepsilon L$.*

PROOF OF LEMMA 3.1. The lemma follows directly from the definitions of $i, f, L$, and $L'$.

## 4. The Structure of $L'$

In the previous section a motivation for the study of $L'$ was provided. $L'$ is just $i^{-1}$ of $L$ and $L$ in turn describes the limiting motions of $S$ as it approaches $C$. In order to study $L'$ we consider (3.1) on $T_2$ itself. We obtain the following system on $T_2$:

$$(4.1) \quad \begin{aligned} d\theta'/dt &= 1 \\ d\sigma'/dt &= 0 \\ du'/dt &= B_0(u_1', u_2', \theta'). \end{aligned}$$

This system only admits entirely well behaved trajectories. $L'$ is closely related to (4.1). We note the following Lemmas.

LEMMA 4.1. *$L'$ is a compact connected subset of $T_2$. $L'$ consists of trajectories of* (4.1) ((4.1) *has no singular points*).

PROOF OF LEMMA 4.1. The claimed properties of $L'$ are proved the same way in which they are proved for the limit set of a trajectory which is contained in the bounded part of the plane. See for instance [5].

LEMMA 4.2. *Let $A$ and $B$ be disjoint open sets of $T_2$. Suppose there exists a trajectory of* (4.1) *which is entirely contained in $A \cap L'$ and a point which belongs* *to $B \cap L'$ then there exist points $p$ and $p'$ in $(T_2 - A - B)$ such that the positive semi-characteristic through $p$ and the negative semi-characteristic through $p'$ remain in $(T' - B) \cap L'$.*

PROOF OF LEMMA 4.2. This lemma for the case where a solution curve approaches a sphere rather than a torus is proved in the previously cited paper of Gomory [2]. The proof, there, makes use of none of the properties which distinguish a sphere from a torus. Hence, no additional proof is required.

We need an additional lemma which will say roughly that if $Z$ is a limit cycle of (4.1) then solution curves of (3.1) sufficiently near $T_2$ can only cross above $Z$ in only one direction. To state this more precisely we need the following definition.

DEFINITION. If $Z$ is a limit cycle of (4.1) the set $B$ will be called a *one sided $\delta$-barrier above $Z$* $\Leftrightarrow$.

(a) $B$ is contained in the part of $E_3'$ for which the $\sigma' - u' - \theta'$ coordinate system is valid and $B$ is homeomorphic to the product of a circle with the closed unit interval $[S_1 \times I]$.

(b) if $g$ is the map of $[S_1 \times I]$ onto $B$ and $t$ and $\psi$ are the natural coordinates on $S_1 \times I$

$$(1) \qquad g[(t, \psi) \mid t = 0] \text{ is just } Z$$

$$(2) \qquad \sigma'\{g[(t, \psi) \mid t > 0]\} > \varepsilon$$

(i.e. the image of all points with $t > 0$ is outside $T_2$)

$$(3) \qquad \sigma'\{g[(t, \psi) \mid t = 1]\} > \varepsilon + \delta.$$

(c) trajectories of (3.1) can cross $B$ only in one direction.

DEFINITION. If $B$ has properties (a) and (b) but trajectories of (3.1) can not cross $B$ at all we shall speak of a *complete $\delta$-barrier above $Z$.*

LEMMA 4.3. *There exists a $\delta(Z)$-barrier above every limit cycle of* (4.1).

PROOF OF LEMMA 4.3. Let $p$ belong to $Z$ and $Q$ be the plane perpendicular to $Z$ through $p$. Let $A$ be a closed line segment in $Q$, outside $T_2$, which starts at $p$ and is perpendicular to $T$. [See Figure 2]. If $A$ is sufficiently small all trajectories through points of $A$ return to $Q$ and the map which sends a point of $A$ into the first intersection of the trajectory through this point with $Q$ is analytic. We

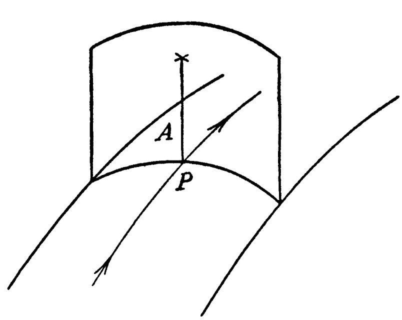


FIG. 2

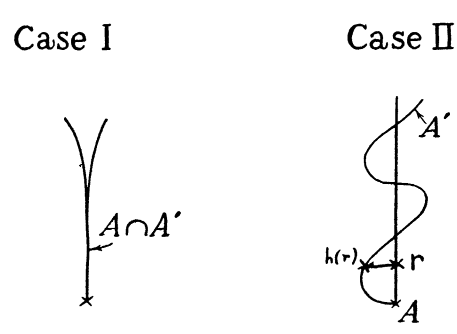


FIG. 3

shall denote this map by $h$ and the image under $h$ of $A$ by $A'$. Since the map $h$ is analytic we only have to consider two possible cases:

CASE I. $A'$ coincides with $A$ in some neighborhood of $p$.

CASE II. $A$ and $A'$ have at most a finite number of intersection points. [See Figure 3]. In Case I there exists a point $r$ of $A$ such that the arcs $[pr]$ of $A$ and $[ph(r)]$ of $A'$ belong to the neighborhood of $p^{i'}$ which $A$ and $A'$ coincide. The closed arc $[rh(r)]$ on $C_r^+$ (the positive semi-characteristic through $r$) is at a finite distance $\delta > 0$ from $T_2$. The band of solutions starting on the closed arc $[pr]$ of $A$ and ending on the closed arc $[ph(r)]$ of $A$ thus form a complete $\delta$-barrier above $Z$.

In Case II let $r$ be a point of $A$ which precedes the first intersection point of $A$ and $A'$ on $A$. It is clear that $r$ can be chosen in such a way that in addition $h(r)$ precedes the first intersection point of $A$ and $A'$ on $A'$. Again the arc $[rh(r)]$ on $C_r^+$ is at a finite distance $\delta > 0$ from $T_2$. The band of trajectories starting on the closed arc $[pr]$ of $A$ and ending on the closed arc $[ph(r)]$ of $A'$ can be closed by the part of the plane $Q$ which is bounded by the arc $[pr]$ of $A$, the arc $[ph(r)]$ of $A'$, and the straight line segment $rh(r)$. It is clear that for $r$ sufficiently close to $p$ trajectories can cross this band only in one direction. Thus a one-sided $\delta$-barrier has been constructed above $Z$.

REMARK. For every limit cycle, $Z$, of (4.1) there exists thus a $\delta(Z)$-barrier. Since $S'$ tends to $T_2$ there exists $t_0(Z)$ such that $S'$ remains within $\delta(Z)$ of $T_2$ for $t > t_0$. We shall say that $S'$ crosses over $Z$ if $S'$ crosses the barrier above $Z$ at a time greater than or equal to $t_0(Z)$.

REMARK ON ANALYTICITY. While Lemmas 4.1 and 4.2 are also true for non-analytic systems provided only the right hand side of the differential equation is differentiable a sufficient number of times, Lemma 4.3 rests strongly on the analytic nature of our problem. If the system would not be analytic the map $h$ would not be analytic which fact, of course, invalidates the whole proof.

## 5. The Structure of $L$ and $L'$

We are now in a position to analyze the various possible structures of $L'$ and thus draw conclusions about the nature of $L$.

CASE A: $B_0 \equiv 0$: (For the definition of $B_0$ we refer to equation (3.1).)

We note that $B_0$ is always zero if $Y_1' \equiv Y_1^2 \equiv 0$ (see equation (2.4)) and may equal zero in some of the other cases.

It follows from (4.1) that if $B_0 \equiv 0$ the trajectories on $T_2$ are just the closed curves $u' = u_0'$. By connectedness $L'$, then, consists either of a single closed curve, or a band of closed curves, or the whole torus. It is interesting that our analysis up to this point is valid even for non-analytic systems which admit a sufficient number of continuous derivatives. But if the system is analytic it is possible to go even further and rule out the case of a band of closed curves.

LEMMA 5.1. *$L'$ can not consist of a band of closed curve solutions of the form $u' = u_0'$.*

PROOF OF LEMMA 5.1. Suppose $L'$ consists of a band of closed curves but is not all of $T_2$. Let $Z_1$ be a closed curve in the interior of the band and $Z_2$ be a closed curve in the exterior. Let $B_1$ and $B_2$ be the $\delta$-barriers above $Z_1$ and $Z_2$ respectively. Furthermore let $B_1$ be a $\delta_1$-barrier and $B_2$ a $\delta_2$-barrier. Then if $R$ is the region $\varepsilon \leqq \sigma < \min [\delta_1, \delta_2]$ $R$ is cut into two sub-regions $A_1$ and $A_2$ by $B_1 \cup B_2$. Since $Z_2$ is not in $L'$ there exists $t_0'$ such that $C(t)$ does not cut $B_2$ for $t \geqq t_0'$. Hence for $t$ greater than some $t_0''$ $C(t)$ is always in $R - B_2$. Since there are points of $L'$ in both $A_1$ and $A_2$ $C(t)$ must be able to go from $A_1$ to $A_2$ and from $A_2$ to $A_1$. But since $B_1$ is a one-sided barrier this is impossible. This contradiction establishes the lemma.

We can now use Lemma 3.1 to translate the above analysis for $L'$ into a result for $L$ and obtain the following theorem.

THEOREM I. *In Case* A $L'$ *is either a single closed curve or the whole torus* $T_2$. *Hence in Case* A, $S$ *approaches* $C$ *in one of two ways: either* $S$ *has a single limiting direction in every surface of section of* $C$ *or every direction in every surface of section is a limiting direction for* $S$.

CASE B: $B_0 \not\equiv 0$. *The rotation number of* (4.1) *is irrational.*

In this case if $p$ is any point of $T_2$ the whole torus belongs to the positive limit set of the trajectory through $p$. It is easy enough to see that if $p$ belongs to $L'$ the positive limit set of the trajectory through $p$ also belongs to $L'$. We again use Lemma 3.1 to relate $L$ to $L'$ and obtain the following theorem.

THEOREM II. *In Case* B *the whole torus* $T_2$ *belongs to* $L'$. *Hence in Case* B *every direction in every surface of section of* $C$ *is a limiting direction of* $S$.

CASE C: $B_p = 0$. *The rotation number of* (4.1) *is rational. In addition* (4.1) *admits an infinity of closed curve solutions.*

In this case by analyticity all solutions are closed curves and we have the same situation as in Case A. Hence we obtain Theorem III.

THEOREM III. *In Case* C $L'$ *consists either of a single closed curve trajectory or of the whole torus. Hence* $S$ *approaches* $C$ *in either of two ways: Either* $S$ *has a finite number of limiting directions in every surface of section of* $C$ *(the same number in every section) or every direction in every surface of section is a limiting direction.*

CASE D: $B_0 = 0$. *The rotation number of* 4.1 *is rational.* 4.1 *only admits a finite number of limit cycles. At least one of the limit cycles is stable or unstable.*

We shall show that if (4.1) admits a single stable or unstable limit cycle $L'$ consists of exactly one closed trajectory.

LEMMA 5.2. *If* $Z$ *is a stable or unstable limit cycle of* $L'$ *then* $Z = L'$.

PROOF OF LEMMA 5.2. Suppose there exists a point $p$ belonging to the compliment of $Z$ in $L'$. We assume that $Z$ is stable. Then there exists an open neighborhood $O$ of $Z$ such that no negative trajectory through a point of $O - Z$ remains in $O$.

Let $F$ be a closed set such that

(a) $$\{Z\} \subset F \subset 0$$

(b) $$p \in F^c \text{ (the complement of } F \text{ on } T_2).$$

Finally let $U$ be an open set such that $\{Z\} \subset U \subset F \subset 0$. Then $U$ and $F^c$ are disjoint open sets. $U$ contains a complete trajectory of $L'$. $F^c$ contains $p \in L'$. Hence by Lemma 4.2 there exists $p' \in L'$ in $\{T_2 - U - F^c\} \subset 0$ such that the negative semi-characteristic through $p'$ remains in $\{T_2 - F^c\} \subset 0$. This is a contradiction.

LEMMA 5.3. *$Z_1$ and $Z_2$ are two limit cycles of* (4.1) *on $T_2$ which belong to $L'$. If $B_{1,2}$ and $C_{1,2}$ are the two components of $\{T_2 - Z_1 - Z_2\}$ there exist points of $L'$ in both $B_{1,2}$ and $C_{1,2}$.*

PROOF OF LEMMA 5.3. Suppose Lemma 5.3 is false. Say $C_{1,2} \cap L'$ is empty. We shall assume that $Z_1$ is stable on the $B_{1,2}$-side (the converse case is handled identically). Then, there exists an open neighborhood 0 of $Z_1$ such that every negative trajectory through a point of $0 \cap B_{1,2}$ leaves $0 \cap B_{1,2}$. Let $F$ be a closed set such that

(a) $$\{Z_1\} \subset F \subset 0$$

(b) $$\{Z_2\} \subset F^c \text{ (compliment of } F \text{ on } T_2).$$

Finally let $U$ be an open set such that $\{Z_1\} \subset U \subset F \subset 0$. Then $U$ and $F^c$ are open sets of $T_2$. $U$ contains a complete trajectory of $L'$; $F^c$ contains a point of $L'$. Hence by Lemma 4.2 there exists a point $p' \in \{T_2 - U - F^c\} \subset 0$ such that the negative semi-characteristic through $p'$ remains in $\{T_2 - F^c\} \cap L'$. Since $p' \in 0 \cap L'$ $p' \in B_{1,2} \cap 0$, for by hypothesis $C_{1,2}$ contains no points of $L'$. On the other hand since the negative semi-characteristic through $p'$ remains in $\{T_2 - F^c\}$ it remains in $B_{1,2} \cap 0$. This is a contradiction in view of the way 0 was chosen.

LEMMA 5.4. *If $L'$ contains at least two limit cycles of* (4.1) *on $T_2$, say $Z_1$ and $Z_2$, and if $Z$ is any other limit cycle of* (4.1) *on $T_2$ then $Z$ belongs to $L'$.*

PROOF OF LEMMA 5.4 Suppose $Z$ does not belong to $L'$. Let $B_{1,2}$ and $C_{1,2}$ have the same meaning as in the preceding Lemma. We assume that $Z \subset B_{1,2}$. By Lemma 5.3 there exists a point, $p$, of $L'$ in $B_{1,2}$. $p$ is either between $Z$ and $Z_1$ or between $Z$ and $Z_2$. In either case it is not possible for both semi-characteristics through $p$ to tend to the set $\{Z_1\} \cup \{Z_2\}$. Hence there exists a third limit cycle of (4.1) which belongs to $L'$, say $Z_3$. $Z_3$ divides $B_{1,2}$ into $B_{1,3}$ and $B_{2,3}$. $Z$ lies in $B_{1,3}$ or $B_{2,3}$. Let us assume $B_{1,3}$. Hence repeating the argument there is a $Z_4$ in $B_{1,3}$ belonging to $L'$. By repeating we can produce an infinity of limit cycles which is a contradiction. Hence $Z$ must belong to $L'$.

We are now in a position to prove that in Case D, where there exists a stable or unstable limit cycle of (4.1), $L'$ consists of a single limit cycle. If $L'$ contains several limit cycles then by Lemma 5.4 it contains all the limit cycles on $T_2$. By hypothesis in Case D this includes a stable or an unstable limit cycle. But then by Lemma 5.2 $L'$ consists of exactly one limit cycle. Suppose some other trajectory, not a limit cycle belonged to $L'$. Then the limit set of this trajectory would also belong to $L'$. But since there are several limit cycles of (4.1) on $T_2$ all deformable into each other the limit set of a trajectory which is not a limit cycle consists of two limit cycles. Since we assumed $L'$ contained only one limit cycle it follows that $L'$ can not contain any trajectories which are not limit cycles. So $L'$ again consists of a single limit cycle. The following theorem follows.

THEOREM IV. *In Case* D *$L'$ consists of exactly one limit cycle; hence in Case* D *$S$ approaches a finite number of limiting directions in every surface of section of $C$ (the number is independent of the section).*

CASE E: *(The Exceptional Case)* $B_0 \not\equiv 0$. *The rotation number of* (4.1) *on $T_2$ is rational. The limit set of* 4.1 *on $T_2$ consists of a finite number of limit cycles none of which is stable or unstable.*

It follows from Lemma 5.4 that $L'$ either contains all the limit cycles of (4.1) on $T_2$ or consists of a single limit cycle only.

Suppose there are several limit cycles of (4.1) on $T_2$ and they all belong to $L'$. If $Z_1$ and $Z_2$ are two successive limit cycles the trajectories of (4.1) through points between $Z_1$ and $Z_2$ tend in one direction to $Z_1$ and in the other direction to $Z_2$. Then by Lemma 5.3 at least one of the points and hence one of the trajectories between $Z_1$ and $Z_2$ belongs to $L'$. In line with the previous results it would have been desirable to show that if $L'$ consists of more than one closed curve solution all of $T_2$ and in particular all points between $Z_1$ and $Z_2$ belongs to $L'$. But there seems to be no reason why an open set between $Z_1$ and $Z_2$ may not be outside $L'$.

There remains a rather special subcase of Case E, where the limit set of (4.1) consists of a single stable-unstable limit cycle. Then this limit cycle certainly belongs to $L'$. But again there may be some other trajectories which belong to $L'$ without all of $T_2$ being contained in $L'$. Theorem V follows.

THEOREM V. *In Case* E, *$L'$ either consists of a single closed curve trajectory or of the whole torus $T_2$ or of all the limit cycles of* (4.1) *together with at least one spiraling trajectory between any two limit cycles. Hence in Case* E *three possibilities arise for the behavior of $S$ near $C$:*

(a) *$S$ tends to a limiting direction in every surface of section of $C$.*

(b) *Every direction in every surface of section of $C$ is a limiting direction.*

(c) *In each surface of section of $C$ there are an infinite number of directions which are limit directions of $S$ and an infinite number of directions which are not limit directions of $S$.*

The results of Theorems I–V can be summarized in the following fashion.

THEOREM VI. *Suppose we are given the differential equation $dX/dt = F(X)$ where $F$ and $X$ are 3-vectors and where the components of the right hand side are analytic functions of the components of $X$. Suppose in addition that this system admits a limit cycle $C$ and a trajectory $S$ which tends towards this limit cycle. Then except for an exceptional case (See below) $S$ approaches $C$ in one of two ways: Either $S$ has a fixed finite number of limiting directions in every surface of section of $C$ or every direction in every surface of section of $C$ is a limit direction of $S$.*

The exceptional case is the case where $B_0 \not\equiv 0$ (See equation (2.4)), the rotation number of (4.1) is rational, and where in addition to these two conditions the limit set of (4.1) on $T_2$ consists of a finite number of closed curve trajectories all of which are semi-stable. A closer description of this case was contained in the discussion of Case E above.

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