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R. E. Gomory[1]

Faces of an Integer Polyhedron[2]

In [5] a connection was given between the integer and non-integer solutions to the linear programing problem

$$(1)\qquad \begin{aligned} &\text{maximize}\, z = cx \\ &Ax = b, \qquad x \geqq 0. \end{aligned}$$

In (1) x is an $m + n$ vector, b is an integer m-vector, c an $m + n$ vector, and A an $m \times (m + n)$ integer matrix containing an $m \times m$ identity matrix. A is assumed to be rearranged and partitioned into an $m \times n$ optimal basis matrix B for the noninteger problem and a collection of nonbasic columns forming the matrix N with $A = (B, N)$. An alternative form of (1) that is useful here for geometric interpretation is to revert to inequalities, writing A a (A', I). Then (1) becomes

$$(1a)\qquad \begin{aligned} &\text{maximize}\, z = c'x' \\ &A'x' \leqq b \end{aligned}$$

where x' and c' are n-vectors.

Under suitable conditions, given in [5], the integer solution to (1) could be obtained from the noninteger one by solving the optimization problem

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$$\max \sum_{i=1}^{i=n} c_i^* t_i$$

subject to the conditions

$$\sum_{i=1}^{i=n} g_i t_i = g_0 \tag{2}$$

where the t_i are required to be nonnegative integers. The c_i^* (which are not important here) are the relative cost coefficients associated with the columns of N, and g_i is the element of the factor module $g = M(I)/M(B)$ corresponding to the ith column of N. Here $M(I)$ is the module of all integer m-vectors, and $M(B)$ the module generated over the integers by the columns of B.

The connection between integer and noninteger solutions established by (2) held only under certain conditions. One way to develop this approach into a general integer programing algorithm would be to develop form (2) new inequalities or "cutting planes" for a method similar to that of [6]. The geometrical interpretation of the solutions to (2) suggests that this is possible and this approach is outlined here.

To see this, consider the cone P' in the space of the variables x' of (1a) formed by using only the inequalities corresponding to the nonbasic variables or equivalently, P' is the cone obtained if in (1a) the nonnegativity condition for the basic variables is dropped. Within P' is the polyhedron P'' which is the convex hull or the integer points of P'. P'' is an interesting object of study in itself. In addition, its faces clearly provide the strongest inequalities or cutting planes for the general integer programing problem that can be deduced locally, i.e. without using the nonlocal information available from the nonnegativity condition on the basic variables.

Since the variables t of (2) determine a corresponding x satisfying the equations of (1) by $t \to x = (B^{-1}(b - Nt), t)$ and hence also determines the x' of (1a), inequalities on the t_i yield inequalities on the x', and in this sense one can talk about an inequality on the t_i being a face of P''.

Faces to P'' can be characterized by the following easily proved theorem which allows their computation by linear programing.

THEOREM 1. *The inequality* $\sum \pi_i t_i \geqq \pi_0$ *is a face of* P'' *if and only if the* π_i *are a basic feasible solution of the system of inequalities,*

(3) $$\pi T \geqq \pi_0$$

made up from all vectors $T = (t_i, \cdots, t_n)$, *satisfying the equations of* (2).

A number of remarks can be made about the feasibility of this computation.

First. Although there are an infinity of T satisfying (2) and hence an infinity of inequalities, it is easy to reduce this to a finite number by considering only the irreducible T, i.e., those T not containing as a vector $T' = (t_i, \cdots, t_n)$ for which $\sum t_i g_i = 0$. Or alternatively one can work only with those T that satisfy (2) for some nonnegative c_i^*.

Second. The trivial faces of P'', those that are simply faces of the original problem (1), can be discarded by choosing $\pi_0 \neq 0$.

Third. The multiplicity of rows in (3) can be dealt with by a row-generating method similar to the methods of [1], [3], [4], and [7]. Because of this, an $n \times n$ basis matrix is the most that is required at any time. The needed row at each simplex step can be generated (at worst) by solving a problem approximately equivalent to (2), essentially a shortest path problem over the group G.

Fourth. There is a simple way of getting a first feasible solution to (3), and hence a face of P'', by solving a single problem like (2) with an additional side calculation that less than doubles the work. (For an estimate of the work involved in solving (2), see [5].) This calculation will not be described here.

Fifth. Duplication, i.e. many columns of N mapping into the same group element, can easily be taken care of and simply reduces the size of the problem to be dealt with.

In addition to providing a method of computing faces of P'' by linear programing, equations (3) lead to the proof of the following theorems which involve considering the tree of shortest paths over the group G.

Let $g_1, \cdots, g_n$ be the group elements corresponding to the columns of N. Choose from among the g_i a basis (we can assume it is $g_1, \cdots, g_p$) so that $G = g_1 \oplus g_2 \oplus \cdots \oplus g_p$, the direct sum. The remaining g_i and the element g_0 corresponding to b can then be represented as p-vectors with respect to this basis; so $g_k = \sum_{i=1}^{i=p} \gamma_{k,i} g_i$ and the following theorem, whose proof is not given here, holds.

THEOREM 2. *If for some basis* $g_1, \cdots, g_p$, *the representation of*

g_0, $g_0 = \sum \gamma_{0,i} g_i$ has a component s for which

$$\gamma_{0,s} \geqq \max_{k>p} \gamma_{k,s},$$

then the π_i given by

$$\pi_0 = \gamma_{0,s}, \quad \pi_s = 1, \quad \pi_k = \gamma_{k,s} k > p, \quad \textit{and} \ \pi_k = 0$$

otherwise give a face of P''.

The condition of the theorem is always met whenever, for some component s, $\gamma_{0,s}$ is exactly one less that the order of g_s. In particular, we have the

COROLLARY. *If g is the direct sum of cyclic groups of order 2, then any basis of g satisfies the conditions of Theorem 2.*

It is easily shown that all A consisting of columns with at most two nonzero entries that are restricted to be 1 or -1, yield groups G that are direct sums of cyclic groups of order 2. This connects with the work of Edmonds [2].

In the next theorem we refer directly to the shortest path tree. In a graph, if any unambiguous method of breaking ties among paths is used so that there is a unique shortest path between two points, the shortest paths from one point to all the others will form a spanning tree. However, different tie-breaking methods produce different trees. If the elements of g are taken as nodes of a graph, and if the g_i, $i = 1, \cdots, n$, are taken as directed arcs connecting each g' to the point $g' + g_i$, we have a graph and hence for any choice of π_i, $i = 1, \cdots, n$, a shortest path tree.

In what follows, b' denotes a right-hand side in (1) and g' is the group element fb' under the natural mapping f which sends $M(I)$ onto G. We can now state Theorem 3.

THEOREM 3. *If R is a shortest path tree for the π_i forming a face of P'', and if $g' = fb'$ is separated from $\bar{0}$ in R by g_0, then the π_i are also a face for the polyhedron P'' resulting from the right-hand side b'.*

The π_0' corresponding to b' can be obtained by adding the tree distance from g_0 to g' to the original π_0.

Finally we state a theorem that allows the computation of faces on a once-and-for-all basis, independent of the particular columns present in the matrix M.

Let the faces F_j be all faces of the higher-dimensional integer polyhedron P^* obtained form (2) by letting the index i in (2) range over all group elements.

$$F_j = (\pi_{0,i}, \pi_{1,i}, \pi_{2,i}, \cdots, \pi_{D-1,i}). \tag{4}$$

THEOREM 4. *The faces F_j', obtained by deleting from (4) all π_{ij} whose corresponding group element is not a column of N, include all faces of the original polyhedron P''.*

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THOMAS J. WATSON RESEARCH CENTER
YORKTOWN HEIGHTS, NEW YORK