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The Group Problems and Subadditive Functions

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1. Introduction

A. Inequalities based on the integer nature of some or all of the variables are useful in almost any algorithm for integer programming. They can furnish cut-offs for branch and bound or truncated enumeration methods, or cutting plane methods. In this paper we describe methods for producing such inequalities.

We will attempt to outline our general approach, taking the pure integer case first.

Consider a pure integer problem

(1) $$Ax = b \; , \quad x \geq 0$$

in which A is an $m \times (m+n)$ matrix, x is an integer $m+n$ vector, and b an m-vector. If we consider a basis B (in most applications this will be an optimal basis) we can write (1) as

$$Bx_B + Nx_N = bx_B \geq 0 \; , \quad x_N \geq 0$$

where x_B is the m-vector of basic variables and x_N the non-basic n-vector. The usual transformed matrix [1, pages 75-80] corresponds to the equations

$$x_B + B^{-1}Nx_N = B^{-1}b \ , \ x_B \geq 0 \ , \ x_N \geq 0 \ , \quad \text{or}$$

(2) $$y + N'z = b' \ , \quad y \geq 0 \ , \ z \geq 0 \ .$$

Taking the i^{th} row we have

$$y_i + \sum_{j=1}^{j=n} n'_{i,j} z_m = b'_i \ .$$

We can form a new but related equation by reducing all coefficients modulo 1 and replacing the equality by equivalence modulo 1. This yields

(3) $$\sum_{j=1}^{j=n} \mathfrak{F}(n'_{ij}) z_j \equiv \mathfrak{F}(b'_i) \qquad (\text{mod } 1) \ .$$

Now any integer vector (y, z) satisfying (2) automatically satisfies (3), so that any inequality

$$\sum_{j=1}^{j=n} \pi_j z_j \geq \pi_0 \ , \quad \text{or} \quad \pi \cdot z \geq \pi_0$$

which is satisfied by all solutions z to (3) is also satisfied by all solutions to (2), i.e.

$$(0, \pi) \cdot (y, z) \geq \pi_0$$

holds for any integer vector (y, z) satisfying (2).

The approach of this paper is to develop inequalities valid for all solutions to (2) by obtaining those valid for all solutions to the simpler equations like (3).

More generally, we can proceed as follows, let ψ be a linear mapping sending the points of m-space into some other topological group S with addition. If we have an equation (like (2))

(4) $$\sum_j C_j x_j = C_0$$

in which the C_j and C_0 are m vectors, we can obtain a new equation by using the mapping ψ to obtain, by linearity,

(5) $$\sum_j \psi(C_j x_j) = \psi(C_0)$$

which is an equation involving a set of group elements in S, the elements $\psi(C_j x_j)$. For integer x_j, $\psi(C_j x_j) = \psi(C_j) x_j$, so equivalent group equations are

(6) $$\sum_j \psi(C_j) x_j = \psi(C_0) .$$

In the discussion leading up to equation (3) the C_j were the columns of the matrix (I, N') and ψ was the mapping that sends an m vector into the fractional part of its i^{th} coordinate. The group S was the unit interval with addition modulo 1. Equation (3) was the equation (6).

Again, if $\pi \cdot x \geq \pi_0$ holds for all integer x satisfying (5) or (6) it holds for integer x satisfying (4).

In this paper we study equations such as (6) and develop inequalities for their solutions which are then satisfied by the solutions to (4). Specifically we study the case where S is I, the unit interval mod 1, and develop inequalities for the equations:

(7) $$\sum_{u \in U} u t(u) = u_0$$

where U represents the set $\psi(C_j) \in I$ and $t(u)$ is a non-negative integer. Equation (7), which we refer to as the problem (or equation), $P(U, u_0)$, is merely (6) rewritten in a different notation.

Returning to equation (4) when some of the x_j are not restricted to be integer, a linear mapping ψ still gives another equation (5) satisfied by all solutions to (4). Thus, any solution to (4) satisfies the equation

$$\sum_j \psi(C_j x_j) = \psi(c_0) .$$

Just as before, if any x_j is required to be integer, then $\psi(C_j x_j) = \psi(C_j) x_j$. Let J_1 denote the subset of j for which x_j is required to be integer and J_2 be the j for which x_j is only required to be non-negative. Then, any solution to (4) with x_j integer for $j \in J_1$ satisfies

$$(8) \qquad \sum_{j \in J_1} \psi(C_j)x_j + \sum_{j \in J_2} \psi(C_j x_j) = \psi(C_0) .$$

When ψ is the same (fractional) map used to derive (3), we rewrite (8) as

$$(9) \qquad \sum_{j \in J_1} \mathfrak{F}(n'_{ij})z_j + \sum_{j \in J_2} \mathfrak{F}(n'_{ij}z_j) \equiv \mathfrak{F}(b'_i) \qquad (\text{mod } 1).$$

Consider $n_{ij}z_j$ for $j \in J_2$. If $n'_{ij} = 0$, then z_j does not really enter into the equation. If $n'_{ij} \neq 0$ we can rescale z_j by letting

$$z'_j = |n'_{ij}|z_j , \quad j \in J_2 .$$

Let $J_2^+ = \{j \in J_2 : n'_{ij} > 0\}$ and $J_2^- = \{j \in J_2 : n'_{ij} < 0\}$. Then $z'_j = n'_{ij}z_j$ for $j \in J_2^+$ and $-z'_j = n'_{ij}z_j$ for $j \in J_2^-$. The restriction $z_j \geq 0$ is equivalent to $z'_j \geq 0$. Hence, (9) becomes

$$(10) \qquad \sum_{j \in J_1} \mathfrak{F}(n'_{ij})z_j + \sum_{j \in J_2^+} \mathfrak{F}(z'_j) - \sum_{j \in J_2^-} \mathfrak{F}(z'_j) \equiv \mathfrak{F}(b'_i) \quad (\text{mod } 1).$$

Since

$$\sum_{j \in J_2^+} \mathfrak{F}(z'_j) \equiv \mathfrak{F}\Big(\sum_{j \in J_2^+} z'_j\Big) \qquad (\text{mod } 1)$$

(10) can be simplified to

$$(11) \qquad \sum_{j \in J_1} \mathfrak{F}(n'_{ij})z_j + \mathfrak{F}(z^+) - \mathfrak{F}(z^-) \equiv \mathfrak{F}(b'_i) \quad (\text{mod } 1)$$

where

$$z^+ = \sum_{j \in J_2^+} z'_j ,$$

$$z^- = \sum_{j \in J_2^-} z'_j .$$

We can rewrite (11) in a form similar to (7) to obtain the problem we call $P^{+}_{-}(U, u_0)$:

$$\sum_{u \in U} ut(u) + \mathfrak{F}(s^{+}) - \mathfrak{F}(s^{-}) = u_0 \ . \tag{12}$$

In this paper, we concentrate on the development of valid inequalities for equations of the form (7) and (12). These inequalities, satisfied by every solution to (7) or (12), are immediately appliciable to the original problem (4). In the case of an inequality

$$\sum_{j \in J_1} \pi_j z_j + \pi^{+} z^{+} + \pi^{-} z^{-} \geq 1 \tag{13}$$

satisfied by every solution to (11), the inequality

$$\sum_{j \in J_1} \pi_j z_j + \sum_{j \in J_2^{+}} (\pi^{+} n'_{ij}) z_j + \sum_{j \in J_2^{-}} (-\pi^{-} n'_{ij}) z_j > 1 \tag{14}$$

is satisfied by every solution to (10), and hence to (4).

2. Problem Definition

Let I be the group formed by the real numbers on the interval $[0, 1)$ with addition modulo 1. Let U be a subset of I and let t be an integer-valued function on U such that (i) $t(u) \geq 0$ for all $u \in U$, and (ii) t has a finite support; that is, $t(u) > 0$ only for a finite subset U_t of U.

The notation and definitions above will be used throughout so that t will always refer to a non-negative integer valued function with finite support.

We say that the function t is a solution to the problem $P(U, u_0)$, for $u_0 \in I - \{0\}$, if

$$\sum_{u \in U} ut(u) = u_0 \ . \tag{1}$$

Here, of course, addition and multiplication are taken modulo 1. Let $T(U, u_0)$ denote the set of all such solutions t to $P(U, u_0)$.

Correspondingly, <u>the problem</u> $P_-^+(U,u_0)$ <u>has solution</u> $t' = (t, s^+, s^-)$ satisfying

$$\sum_{u \in U} ut(u) + \mathfrak{F}(s^+) - \mathfrak{F}(s^-) = u_0 \tag{2}$$

where t is, as before, a non-negative integer valued function on U with a finite support, where s^+, s^- are non-negative real numbers, and where $\mathfrak{F}(x)$ denotes the element of I given by taking the fractional part of a real number x. Let $T_-^+(U,u_0)$ denote the set of solutions $t' = (t, s^+, s^-)$ to $P_-^+(U,u_0)$.

It is also possible to define problems $P^+(U,u_0)$ and $P_-(U,u_0)$ in which only s^+ or s^- appear, and these problems are useful in some situations. Their development parallels that of $P_-^+(U,u_0)$.

The notation $u \in I$ will mean that u is a member of the group I so that arithmetic is always modulo 1. If we want to consider u as a point on the real line with real arithmetic, we will write $|u|$. Thus, $|u|$ and $\mathfrak{F}(x)$ are mappings in opposite directions between I and the reals. And, in fact, $\mathfrak{F}(|u|) = u$ but x and $|\mathfrak{F}(x)|$ may differ by an integer.

<u>Definition I.</u> <u>Valid Inequalities.</u> For any problem $P(U,u_0)$, we have so far defined the solution set $T(U,u_0)$. A <u>valid inequality for the problem</u> $P(U,u_0)$ is a real-valued function π defined for all $u \in I$ such that

$$\pi(u) \geq 0\ , \quad \text{all } u \in I\ , \quad \text{and} \quad \pi(0) = 0\ , \tag{3}$$

and

$$\sum_{u \in U} \pi(u)t(u) \geq 1\ , \ \text{all } t \in T(U,u_0)\ . \tag{4}$$

For the problem $P_-^+(U,u_0)$, $\pi' = (\pi, \pi^+, \pi^-)$ is a <u>valid inequality for</u> $P_-^+(U,u_0)$ when π is a real-valued function on I satisfying (3), and π^+, π^- are non-negative real numbers such that

(5) $$\sum_{u \in U} \pi(u)t(u) + \pi^{+}s^{+} + \pi^{-}s^{-} \geq 1 \text{ , all } t' \in T^{+}_{-}(U, u_0) \text{ .}$$

A valid inequality (π, π^{+}, π^{-}) for $P^{+}_{-}(I, u_0)$ can be used to give a valid inequality for $P(U, u_0)$ or $P^{+}_{-}(U, u_0)$ for any subset U of I. For example, $\Sigma\pi(u)t(u) \geq 1$ is clearly true for any $t \in T(U, u_0)$ since that t can be extended to a function t' belonging to $T(I, u_0)$ by letting $t'(u) = 0$ for $u \in I-U$. Thus, the problem $P^{+}_{-}(I, u_0)$ acts as a master problem as in [2] where the master problem is a group problem with all group elements present. This fact is the main reason for studying the case $U = I$ in such detail. However, the next two properties of valid inequalities do not necessarily carry over to subsets U and I.

Definition 2. Minimal Valid Inequalities. A valid inequality π for $P(U, u_0)$ is a minimal valid inequality for $P(U, u_0)$ if there is no other valid inequality ρ for $P(U, u_0)$ satisfying $\rho(U) < \pi(U)$, where $\rho(U) < \pi(U)$ is defined to mean $\rho(u) \leq \pi(u)$ for all $u \in U$ and $\rho(u) < \pi(u)$ for at least one $u \in U$. A valid inequality π' for $P^{+}_{-}(U, u_0)$ is a minimal valid inequality for $P^{+}_{-}(U, u_0)$ satisfying $\rho'(u) < \pi'(u)$ where $\rho'(u) < \pi'(U)$ is defined to mean

$$\rho^{+} \leq \pi^{+} \text{ , } \rho^{-} \leq \pi^{-} \text{ ,}$$

and

$$\rho(u) \leq \pi(u) \text{ , } u \in U \text{ ,}$$

with strict inequality holding for at least one of the above inequalities.

The minimal valid inequalities are important because a valid inequality which is not minimal is implied by some other valid inequality. Notice that we have scaled the inequalities to have a right-hand side equal to one, and minimality is always with respect to that scaling.

Definition 3. Extreme Valid Inequalities. A valid inequality π for $P(U, u_0)$ is an extreme valid inequality for $P(U, u_0)$ if π can not be written as $\pi = \frac{1}{2}\rho + \frac{1}{2}\sigma$ for $\rho \neq \sigma$ where ρ, σ are valid inequalities for $P(U, u_0)$.

A valid inequality $\pi' = (\pi, \pi^+, \pi^-)$ for $P^+_-(U, u_0)$ is an extreme valid inequality for $P^+_-(U, u_0)$ if π' cannot be written as $\pi' = \frac{1}{2}\rho + \frac{1}{2}\sigma'$ for $\rho' \neq \sigma'$ where ρ', σ' are valid inequalities for $P^+_-(U, u_0)$.

Theorem I.1 of [3] says that the extreme valid inequalities are also minimal. These inequalities are in some sense "the best" possible since they cannot be derived from any other valid inequalities.

Definition 4. Subadditive Valid Inequalities. A valid inequality π for $P(U, u_0)$ is a subadditive valid inequality for $P(U, u_0)$ if

$$\pi(u) + \pi(v) \geq \pi(u+v) \tag{6}$$

whenever all three of u, v, and $u+v$ are in U.

For a valid inequality π' for $P^+_-(U, u_0)$ to be subadditive, we require, in addition to (6),

$$\pi(u) + \pi^+ |v-u| \geq \pi(v), \text{ whenever } u, v, \in U, \ |u| < |v|, \tag{7}$$

$$\pi(u) + \pi^- |u-v| \geq \pi(v), \text{ whenever } u, v, \in U, \ |u| > |v| . \tag{8}$$

Theorems I.1 and I.2 of [3] prove the following sequence of inclusions: the set of valid inequalities include the subadditive valid inequalities which include minimal valid inequalities which include extreme valid inequalities. The subadditive valid inequalities form a convex set contained in the larger convex set of valid inequalities. Theorem I.3 of [3] says that the extreme points of the set of subadditive valid inequalities include all the extreme valid inequalities. Further, among the extreme subadditive valid inequalities, those which are extreme valid inequalities are the minimal ones. This fact allows us to actually find the extreme valid inequalities for some problems.

3. Subadditivity for Subgroups U.

The problems for which we can find extreme valid inequalities are $P(U, u_0)$ or $P^+_-(U, u_0)$ where U is a nonempty subgroup of I. We permit $U = I$ and note that 0 is always in U.

Definition 5. A function π defined on I is subadditive on a subgroup U of I if

$$\pi(u) \geq 0 , \quad u \in I , \quad \pi(0) = 0 , \quad \text{and}$$

$$\pi(u) + \pi(v) \geq \pi(u+v), u, v, \in U .$$

The function π is not assumed to be a valid inequality. Theorem I.5 of [3] establishes the close connection between subadditive functions on U and subadditive valid inequalities. That theorem asserts that the subadditive valid inequalities for $P(U, u_0)$ are precisely the subadditive functions π satisfying $\pi(u_0) \geq 1$. Furthermore, if π is a subadditive function on U and $\pi(u_0) > 0$ for some $u_0 \in U$, then π^* defined by

$$(9) \qquad \pi^*(u) = \frac{\pi(u)}{\pi(u_0)} , \quad u \in I ,$$

is a valid inequality for $P(U, u_0)$.

The analogous theorem for $P^+_-(U, u_0)$ will now be developed.

Definition 6. $\pi' = (\pi, \pi^+, \pi^-)$ is an extended subadditive function on a subgroup U of I if π is subadditive on U, and if, in addition

$$(11) \qquad \pi^+ |u| \geq \pi(u) , \quad u \in U ,$$

$$(12) \qquad \pi^- |u| \geq \pi(-u) , \quad -u \in U .$$

Theorem I.5B of [3] says that the subadditive valid inequalities are precisely the extended subadditive functions which satisfy both:

(13) $\pi(u) + \pi^{+}|u_0 - u| \geq 1$ whenever $u \in U$ and $|u| \leq |u_0|$,

(14) $\pi(u) + \pi^{-}|u - u_0| \geq 1$ whenever $u \in U$ and $|u| \leq |u_0|$,

Although subadditivity of π on I is not easily characterized, a graphical representation can be given. If π is drawn as a periodic function with period 1 as in figure 1, and if we then shift the image of π by transferring the origin to another point $(u, \pi(u))$, as shown in figure 1 by dotting lines, then subadditivity of π is equivalent to the dotted line staying above the solid line. The reason is that the points on the dotted line are of the form $(u + v, \pi(u) + (v))$.

Some examples of subadditive functions are shown in figure 2. Figure 2(a) shows Gomory's fractional cut, figure 2(b) shows Gomory's mixed integer cut, and figure 2(c) is a more complicated function.

When π is subadditive on I, then (π, π^{+}, π^{-}) is an extended subadditive function on I if, and only if, π has a right derivative at 0 and a left derivative at 1, and π^{+} is larger than or equal to the right derivative at 0 while π^{-} is larger than or equal to the absolute value of the left derivative at 1. In figure 2(a), π has a right derivative at 0 equal to 1, but π has no finite left derivative at 1. In both figures 2(b) and (c), π has both left and right derivatives at 0 and 1, so π^{+}, π^{-} could be set to make (π, π^{+}, π^{-}) an extended subadditive function in those two cases.

4. Minimality for Subgroups U.

Theorem 1.6 of [3] is as follows: If U is a subgroup of I with $u_0 \in U$ and if π is a valid inequality for $P(U, u_0)$, then π is a minimal valid inequality if and only if

(15) $$\pi(u) + \pi(u_0 - u) = 1, \quad \text{all } u \in U.$$

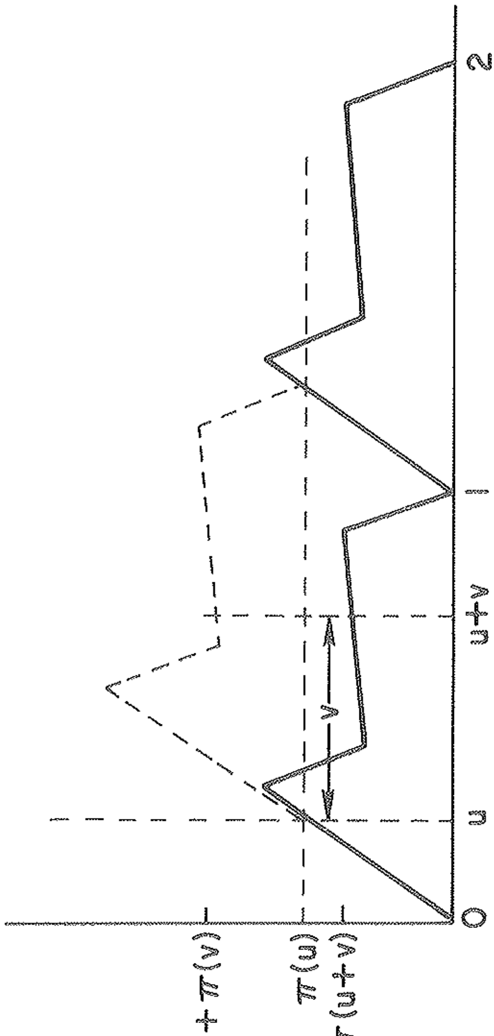

Figure 1

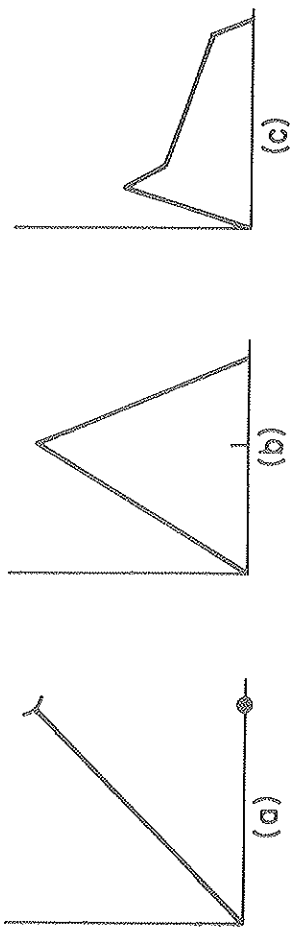

Figure 2

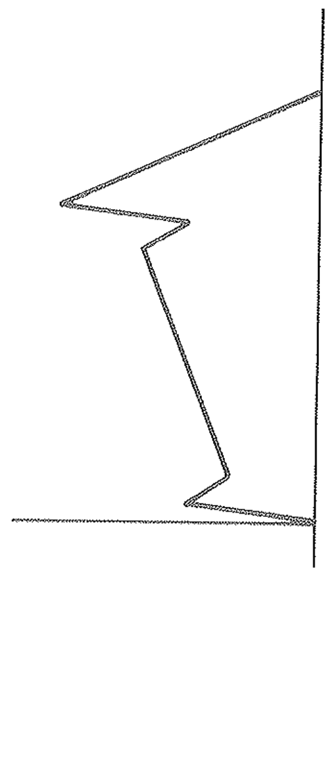

Figure 3

This condition imposes a peculiar symmetry on π so that $\pi(u)$ for $\frac{1}{2}u_0 < u < u_0$ is determined by $\pi(u)$ on $[0, \frac{1}{2}u_0]$, for example. This symmetry is illustrated in figure 3. One way of picturing it is that $\pi(u)$ and $\pi(u_0-u)$ must change by equal amounts but in opposite directions as u increases from 0 or as u decreases from 1 .

5. $P(G_n, u_0)$, $u_0 \in G_n$

Let G_n denote the subset

$$G_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\}$$

of I . The elements of G_n will be denoted $g_i = \mathfrak{F}(i/n)$. Each set G_n for $n \geq 1$ is a subgroup of I . By virtue of G_n being a subgroup, the results of 3 and 4 apply to this section.

The results from 3 and 4 are specialized in Theorem II. 2 of [3]: The extreme valid inequalities for $P(G_n, u_0), u_0 \in G_n$, are the extreme points of the solutions to

$$(16) \qquad \pi(g_i) \geq 0 \ , \quad \pi(0) = 0 \ ,$$

$$(17) \qquad \pi(g_i) + \pi(g_j) \geq \pi(g_i+g_j) \ ,$$

$$(18) \qquad \pi(u_0) \geq 1$$

which satisfy the additional equations,

$$\pi(g_i) + \pi(u_0-g_i) = 1, \qquad g_i \in G_n \ .$$

In particular, (4) implies $\pi(u_0) = 1$ since $\pi(0) = 0$.

In [2], the extreme valid inequalities, or faces, are given for all G_n, $n = 1, \ldots, 11$. In addition, the faces are given for non-cyclic, but still abelian, groups of order less than 11. An example of the linear inequalities defining those faces is given below for $n = 6$ and $u_0 = g_5$:

$\pi(1)$	$\pi(2)$	$\pi(3)$	$\pi(4)$	$\pi(5)$	
1			1	-1	= 0
	1	1		-1	= 0
2	-1				≥ 0
1	1	-1			≥ 0
1		1		-1	≥ 0
	2			-1	≥ 0
-1		1		1	≥ 0
	-1			2	≥ 0

6. $P_{-}^{+}(G_n, u_0)$, $u_0 \in I$

The condition (2) now becomes

$$g_1 t(g_1) + \ldots + g_{n-1} t(g_{n-1}) + \mathfrak{F}(s^+) - \mathfrak{F}(s^-) = u_0 ,$$

where $g_i = \mathfrak{F}(i/n)$ as before and where the $t(g_i)$ must be non-negative integers and s^+, s^- must be non-negative real values. We no longer confine u_0 to be in G_n. Let $L(u_0)$ and $R(u_0)$ denote, respectively, the points of G_n immediately below and above u_0. If u_0 happens to be in G_n, then $L(u_0) = R(u_0) = u_0$.

From Theorem II. 2B of [3] we know that the extreme valid inequalities π' for $P_{-}^{+}(G_n, u_0)$, $u_0 \in I$, are the extreme points of the solutions to the system of linear equations and inequalities (16), (17), and all of the following:

(19) $$\pi^{+} \frac{1}{n} \geq \pi(g_1), \; g_1 = \mathfrak{F}\left(\frac{1}{n}\right) ,$$

(20) $$\pi^{-} \frac{1}{n} \geq \pi(g_n - 1), \; g_n - 1 = \mathfrak{F}((n-1)/n).$$

(21) $$\pi(L(u_0)) + \pi^{+} |u_0 - L(u_0)| = 1 ,$$

(22) $$\pi(R(u_0)) + \pi^{-} |R(u_0) - u_0| = 1 ,$$

(23) for all $g_i \in G_n$, $\pi(g_i) + \pi(L(u_0)-g_i) = \pi(L(u_0))$

or $\pi(g_i) + \pi(R(u_0)-g_i) = \pi(R(u_0))$.

In [3], the extreme valid inequalities for $P_-^+(G_n, u_0)$ are given for $n = 1, \ldots, 7$ and all u_0 . We give below the inequalities used to generate these faces for $n = 6$ and $u_0 = \frac{3}{4}$:

π^+	$\pi(1)$	$\pi(2)$	$\pi(3)$	$\pi(4)$	$\pi(5)$	π^-	
	2	-1					≥ 0
	1	1	-1				≥ 0
	1		1	-1			≥ 0
	1			1	-1		≥ 0
		2		-1			≥ 0
		1	1		1		≥ 0
	-1		1	1			≥ 0
		-1	1		1		≥ 0
		-1		2			≥ 0
			-1	1	1		≥ 0
$\frac{1}{6}$	-1						≥ 0
					-1	$\frac{1}{6}$	≥ 0
$\frac{1}{12}$				1			$= 1$
					1	$\frac{1}{12}$	$= 1$

In addition, condition (23) must be checked in order for (π, π^+, π^-) to represent an extreme valid inequality. In the appendix of [3], the computation and condition (23) are discussed.

7. Valid Inequalities for $P(U, u_0)$

We now connect the results about $P(G_n, u_0)$ with the general problem $P(U, u_0)$. Here, U can be any subset of the unit interval including the interval I itself. Theorem III.1 of [3] says that valid inequalities can be obtained simply by connecting the points $(g_n, \pi(g_n))$ by straight line segments. More precisely, if π is a subadditive function on G_n and if

$$(24) \qquad \pi(u) = n\{ |u - L(u)| \pi(R(u)) + |R(u) - u| \pi(L(u)) \}, \quad u \in I - G_n ,$$

Then, π is a subadditive function on I, and π^* defined on I by

$$\pi^*(u) = \frac{\pi(u)}{\pi(u_0)}, \quad u \in I ,$$

is a valid inequality for any $P(U, u_0)$, U a subset of I, provided $\pi(u_0) > 0$.

8. Valid Inequalities for $P_-^+(U, u_0)$

From valid inequalities for $P_-^+(G_n, u_0)$, a different method for generating valid inequalities for $P_-^+(U, u_0)$ is available. This method will be referred to as the <u>two-slope fill-in</u> (Theorem III. 3 [3]). Let $\pi' = (\pi, \pi^+, \pi^-)$ be an extended subadditive function on G_n. Define $\pi(u)$ for $u \in I - G$ by

$$(25) \qquad \pi(u) = \min\{ \pi(L(u)) + \pi^+ |u - L(u)| , \quad \pi(R(u)) + \pi^- |R(u) - u| \} .$$

Then, π' is an extended subadditive function on I, and ρ' defined by

$$\rho' = \frac{1}{\pi(u_0)} (\pi, \pi^+, \pi^-)$$

is a valid inequality for $P_-^+(U, u_0)$ provided $\pi(u_0) > 0$.

Theorem II. 2B of [3] shows how to compute faces for $P_-^+(G_n, u_0)$ and this theorem shows how to use them to generate valid inequalities for any U. Table 2 of [3] was obtained using Theorem II. 2B, and we will frequently refer to the two-slope fill-in of those faces.

The functions π generated by this two-slope fill-in can also be used for $P(U, u_0)$ as well as $P_-^+(U, u_0)$. That is, for a problem with neither s^+ nor s^-, the π can be used, ignoring π^+ and π^-, to give a valid inequality for $P(U, u_0)$. The functions π generated by the two-slope fill-in have the advantage over the straight line fill-in that they are generated for a particular u_0 so that $\pi(u_0)$ will be large and the resulting inequality stronger.

Example 1: Consider the integer linear program

$$x_j \geq 0\ ,\ x_j \ \text{integer},\quad j = 1, 2, 3, 4, 5$$

$$x_1 + 2x_2 + x_3 + x_4 + 5x_5 = 10$$

$$3x_1 - 3x_2 + 2x_3 - 3x_4 + 3x_5 = 5$$

$$x_1 + x_2 + x_3 + 2x_4 + 2x_5 = Z(\min)\ .$$

The optimum linear programming tableau is

$$x_1 + \frac{7}{9}x_3 - \frac{1}{3}x_4 + 2\frac{1}{3}x_5 = 4\frac{4}{9}$$

$$x_2 + \frac{1}{9}x_3 + \frac{2}{3}x_4 + 1\frac{1}{3}x_5 = 2\frac{7}{9}$$

$$\frac{1}{9}x_3 + 1\frac{2}{3}x_4 + \frac{1}{3}x_5 = Z(\min)\ .$$

The optimum linear programming solution is $x_1 = 4\frac{4}{9}$, $x_2 = 2\frac{7}{9}$, $x_3 = x_4 = x_5 = 0$, $z = 7\frac{2}{9}$. From the first row of the tableau, using as the mapping $\psi : \psi(A^j x_j) = \mathfrak{F}(a_{ij})x_j$, we obtain

$$\frac{7}{9}x_3 + \frac{2}{3}x_4 + \frac{1}{3}x_5 = \frac{4}{9}(\text{mod } 1)\ .$$

That is, $U = \{7/9, 2/3, 1/3\}$, $u_0 = 4/9$. The mapping used in the introduction is simply this; that is, we get a problem $P(U, u_0)$ from every row of an optimum linear programming tableau for which the basic variable is integer constrained but at a fractional value.

From Appendix 5 of [2], we find the following three extreme valid inequalities among those for $P(G_n, u_0)$, $n = 2, 3, 6$.

$$P(G_2, \tfrac{1}{2}):\ \pi_1(0) = 0\ ,\quad \pi_1(\tfrac{1}{2}) = 1\ ;$$

$$P(G_3, \tfrac{1}{3}):\ \pi_2(0) = 0\ ,\quad \pi_2(\tfrac{1}{3}) = 1\ ,\quad \pi_2(\tfrac{2}{3}) = \tfrac{1}{2}\ ;$$

$$P(G_6, \tfrac{1}{2}):\ \pi_3(0) = 0\ ,\quad \pi_3(\tfrac{1}{6}) = \tfrac{1}{3}\ ,\quad \pi_3(\tfrac{2}{6}) = \tfrac{2}{3}\ ,$$

$$\pi_3(\tfrac{3}{6}) = 1\ ,\quad \pi_3(\tfrac{4}{6}) = \tfrac{1}{3}\ ,\quad \pi_3(\tfrac{5}{6}) = \tfrac{2}{3}\ .$$

We could take any of the faces for cyclic groups from Appendix 5 and use them in the following way. The linear interpolation of extends π_1, π_2, π_3 to the interval I:

$$\pi_1(u) = \begin{cases} 2u\ , & 0 \le u \le \frac{1}{2} \\ 2-2u, & \frac{1}{2} < u < 1 \end{cases}$$

$$\pi_2(u) = \begin{cases} 3u\ , & 0 \le u \le \frac{1}{3} \\ \frac{3}{2} - \frac{3}{2}u, & \frac{1}{3} < u < 1 \end{cases}$$

$$\pi_3(u) = \begin{cases} 2u, & 0 \le u \le \frac{1}{2} \\ 3-4u, & \frac{1}{2} < u < \frac{2}{3} \\ -1+2u, & \frac{2}{3} < u \le \frac{5}{6} \\ 4-4u, & \frac{5}{6} < u < 1 \end{cases}$$

Our congruence problem has $u_0 = 4/9$, and so $\pi_1(u_0) = 8/9$, $\pi_2(u_0) = 5/6$, $\pi_3(u_0) = 8/9$. Since $U = \{7/9, 2/3, 1/3\}$, the valid inequalities from π_1, π_2, and π_3 are of the form

$$\frac{\pi_i(\frac{7}{9})}{\pi_i(u_0)}x_3 + \frac{\pi_i(\frac{2}{3})}{\pi_i(u_0)}x_4 + \frac{\pi_i(\frac{1}{3})}{\pi_i(u_0)}x_5 \geq 1 ,$$

for $i = 1, 2, 3$, and are given below:

$$\frac{1}{2}x_3 + \frac{3}{4}x_4 + \frac{3}{4}x_5 \geq 1 ;$$

$$\frac{2}{5}x_3 + \frac{3}{5}x_4 + \frac{6}{5}x_5 \geq 1 ;$$

$$\frac{5}{8}x_3 + \frac{3}{8}x_4 + \frac{3}{4}x_5 \geq 1 .$$

The 'fractional cutting plane' is, here,

$$\frac{7}{9}x_3 + \frac{2}{3}x_4 + \frac{1}{3}x_5 \geq \frac{4}{9}, \text{ or } \frac{7}{4}x_3 + \frac{3}{2}x_4 + \frac{3}{4}x_5 \geq 1 .$$

That inequality is obtained from the subadditive function π on I given by $\pi(u) = u$. The figure 4 illustrates the functions π, π_1, π_2, π_3.

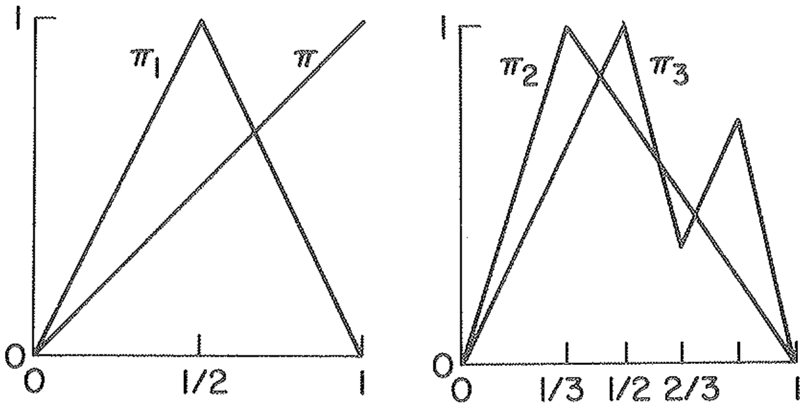

Figure 4

Example 2: Consider the same integer program but without the integrality restriction on x_5. The first row of the optimal linear programming tableau now gives the congruence:

$$\frac{7}{9}x_3 + \frac{2}{3}x_4 + s^+ \equiv \frac{4}{9} \pmod 1$$

where $s^+ = \frac{7}{3}x_5$. Thus, $U = \{\frac{7}{9}, \frac{2}{3}\}$ and $u_0 = \frac{4}{9}$.

From table 2 of the appendix of [3], for $n = 1$ the only extreme valid inequality has

$$\pi^+ = \frac{1}{|u_0|},\ \pi^- = \frac{1}{1-|u_0|}\ .$$

Here, $u_0 = 4/9$ so $\pi^+ = 9/4$ and $\pi^- = 9/5$. Another extreme valid inequality, this time for $n = 3$, is

$$\pi(\frac{1}{3}) = \frac{1}{3|u_0|},\ \pi(\frac{2}{3}) = \frac{1}{6|u_0|},\ \pi^+ = \frac{1}{|u_0|},\ \pi^- = \frac{6|u_0|-1}{4|u_0|-6|u_0|^2}.$$

Since $u_0 = 4/9$ here,

$$\pi(\frac{1}{3}) = \frac{3}{4},\ \pi(\frac{2}{3}) = \frac{3}{8},\ \pi^+ = \frac{9}{4},\ \pi^- = \frac{45}{1.6}\ .$$

The two-slope fill-in extends these two inequalities to functions π_1 and π_2 on the unit interval:

$$\pi_1(u) = \begin{cases} \frac{9}{4}|u| & ,\ 0 \le |u| \le \frac{4}{9} \\ \frac{9}{5}(1-|u|), & \frac{4}{9} \le |u| \le 1, \end{cases}$$

$$\pi_2(u) = \begin{cases} \frac{9}{4}|u|, & 0 \le |u| \le \frac{4}{9}, \\ \frac{3}{8} + \frac{45}{16}(\frac{2}{3} - |u|), & \frac{4}{9} \le |u| \le \frac{2}{3} \\ \frac{3}{8} + \frac{9}{4}(|u| - \frac{2}{3}), & \frac{2}{3} \le |u| \le \frac{7}{9} \\ \frac{45}{16}(1 - |u|), & \frac{7}{9} \le |u| \le 1 \end{cases}$$

Since $U = \{\frac{7}{9}, \frac{2}{3}\}$, a valid inequality is

$$\pi_i(\frac{7}{9})x_3 + \pi(\frac{2}{3})x_4 + \pi_i^+ s^+ \ge 1, \quad i = 1, 2, \qquad \text{or}$$

$$\pi_i(\frac{7}{9})x_3 + \pi_i(\frac{2}{3})x_4 + \frac{7}{3}\pi_i^+ x_5 \ge 1 \quad i = 1, 2.$$

Evaluating π_i at $\frac{7}{9}$ and $\frac{2}{3}$ gives the two valid inequalities

$$\frac{2}{5}x_3 + \frac{3}{5}x_4 + \frac{21}{4}x_5 \ge 1, \quad \text{and}$$

$$\frac{5}{8}x_3 + \frac{3}{8}x_4 + \frac{21}{4}x_5 \ge 1.$$

Other inequalities can be generated in the same way from Table 2 of [3].

Example 3: Consider the integer program from Example 1, but let us use the functions π_1, π_2 from Example 2 to give cutting planes for that pure integer program. This example will provide a comparison on the coefficient of a variable (here x_5) which is an integer variable in one case and a continuous variable in another.

Now $U = \{\frac{7}{9}, \frac{2}{3}, \frac{1}{3}\}$ as in Example 1, and the valid inequalities from π_1 and π_2 given in Example 2 are:

$$\frac{2}{5}x_3 + \frac{3}{5}x_4 + \frac{3}{4}x_5 \ge 1, \qquad \text{and}$$

$$\frac{5}{8}x_3 + \frac{3}{8}x_4 + \frac{3}{4}x_5 \geq 1 .$$

Notice that the coefficients for x_5 are smaller here than in Example 2, illustrating the additional strength gained by the integrality assumption on x_5.

Figure 5 shows π_1 and π_2 used here.

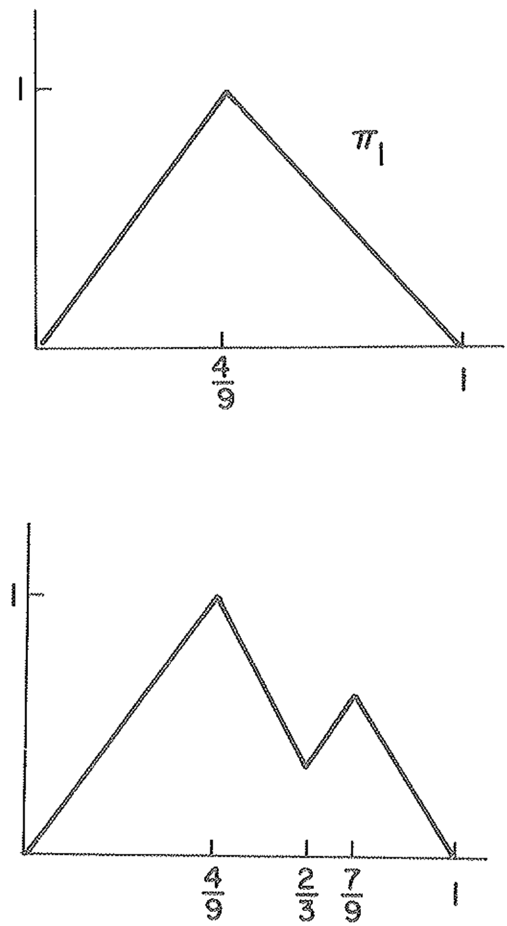

Figure 5

9. Rounding Methods

This section will give some ways to approximate the answer to a given cyclic group problem by a smaller group. For any problem:

$$x_j \geq 0 \text{ and integer}, \quad s^+ \geq 0, \; s^- \geq 0$$

$$\sum_{j=1}^{n} f_j x_j + s^+ - s^- \equiv f_0 \pmod 1$$

$$\sum_{j=1}^{n} c_j x_j + c^+ s^+ + c^- s^- = z ,$$

form the points (f_j, c_j) in the plane. The rounding methods are based on forming a function below the points and with no slope greater than c^+ nor less than $-c^-$. This function is then lowered to one which is subadditive by solving a smaller cyclic group problem. The derived function does give a valid inequality for the original group problem, and the value of the function at f_0 is a lower bound on the value of the cost z to satisfy the group problem.

When s^+ (or s^-) is not present in the problem, the restriction that no slope be greater than c^+ (or less than c^-) can be dropped.

The simplest method is first shown when neither s^+ nor s^- is present. In that case, there are no restrictions on the slopes. The method is described below. The number H can be any positive integer. Define

$$\gamma_h = \min_j \{c_j : \frac{h-1}{H} < f_j < \frac{h+1}{H}\}$$

for $h = 1, 2, \ldots, H-1$. Let $\gamma_0 = \gamma_H = 0$. Now, solve the cyclic group problem

$$\text{minimize} \sum_{h=1}^{H-1} \gamma_h t_h$$

$$\sum_{h=1}^{H-1} h\, t_h \equiv h_0 \pmod H$$

$$t_h \geq 0 \text{ and integer,}$$

where h_0 is either the integer above or below Hf_0.

Shortest path methods, including the one giving in Section 5, will give numbers $d(h) \leq \gamma_h$ such that

$$d(h) + d(g) \geq d(h + g \pmod H) .$$

Extend d to a function ρ on the unit interval by letting

$$\rho(x) = \max \{d(\lceil Hx \rceil), d(\lfloor Hx \rfloor)\}$$

where $\lceil y \rceil$ means the smallest integer larger than or equal to y and $\lfloor y \rfloor$ means the largest integer less than or equal to y. Figure 5 illustrates this function. The claim is that

$$\sum_{j=1}^{n} \rho(f_j)\, x_j \geq \rho(f_0)$$

is a valid inequality for the original problem. Furthermore, $\rho(f_0)$ is a lower bound on z for the original problem, and $\rho(f_0) \geq d(h_0)$.

The original choice of γ_h assures that $\rho(f_j) \leq c_j$. Thus, if the inequality is valid, the bound of $\rho(f_0)$ is true. All that remains is to show that ρ is subadditive, that is, $\rho(x) + \rho(y) \geq \rho(x+y \pmod 1)$. But,

$$\rho(x) + \rho(y) = \max \{d(\lceil Hx \rceil), d(\lfloor Hx \rfloor)\}$$
$$+ \max \{d(\lceil Hy \rceil, d(\lfloor Hy \rfloor)\} ,$$

and $\rho(x+y(\text{mod}))$ is either

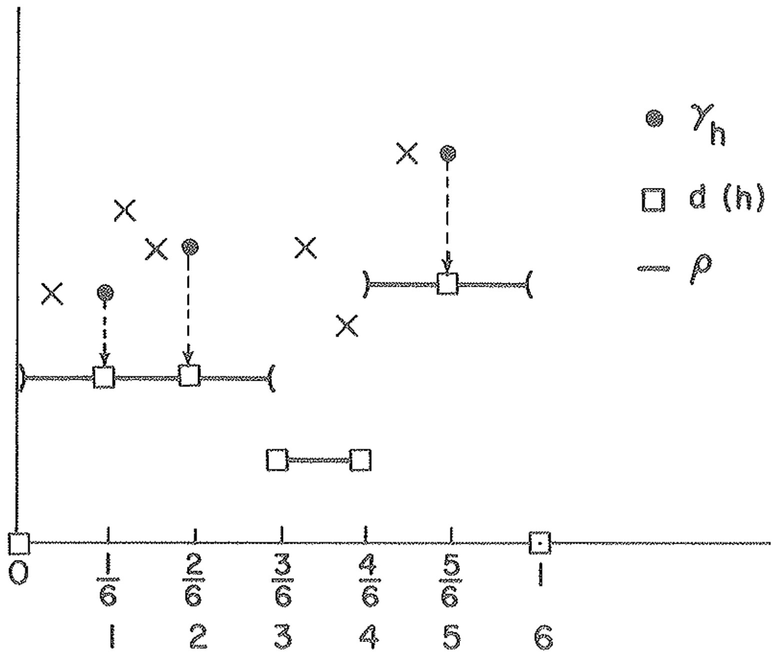

Figure 6

$$\max\{d(\lceil Hx\rceil + \lceil Hy\rceil),\ d(\lceil Hx\rceil + \lfloor Hy\rfloor)\}$$

or

$$\max\{d(\lceil Hx\rceil + \lfloor Hy\rfloor),\ d(\lfloor Hx\rfloor + \lfloor Hy\rfloor)\}\ ,$$

where the $+$ here is mod H. Which of the two terms is $\rho(x+y(\text{mod } 1))$ depends on where $Hx+Hy(\text{mod } H)$ happens to fall relative to $\lfloor Hx\rfloor + \lfloor Hy\rfloor$, $\lceil Hx\rceil + \lfloor Hy\rfloor$, $\lceil Hx\rceil + \lceil Hy\rceil$. However, any one of the numbers $d(h_1+h_2)$, for $h_1 = \lfloor Hx\rfloor$ or $\lceil Hx\rceil$ and $h_2 = \lceil H_y\rceil$ or $\lceil H_y\rceil$, is less than or equal to $\rho(x) + \rho(y)$ because that sum is the sum of two maxima including $d(h_1)$ and $d(h_2)$.

This method should help to establish the general principle. The general method involves extending the numbers $d(h)$, $h = 0, \ldots, H$, to a function $\rho(x)$, $0 \le x \le 1$, such that $\rho(h/H) = d(h)$. Further, the function ρ should be subadditive on the whole interval $[0,1]$ whenever d is subadditive on $0, 1, \ldots, h$. Three general forms of such extensions are given below.

Given values $0 = d(0), d(1), \ldots, d(H-1), D(H) = 0$, and $d^+ > 0$, $d^- > 0$, such that $d(1) \le d^+/H$ and $d(H-1) \le d^-/H$, let $L(x) = \lfloor Hx \rfloor$ and $R(x) = \lceil Hx \rceil$ for $0 \le x \le 1$. Define

$$\rho_1(x) = \min\{d(L(x)) + d^+(x - L(x)),\ d(R(x)) + d^-(R(x) - x),$$

$$\max\{(L(x)),\ d(R(x))\}\} .$$

Here, the d are assumed subadditive, i.e., $d(h) + d(g) \ge d(g+h \pmod H))$.

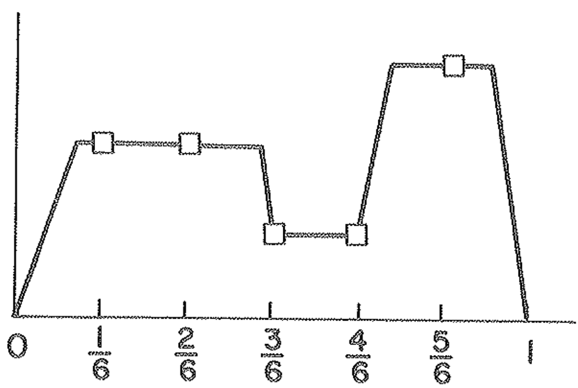

Figure 7

In order for ρ_1 to be continuous, $d(h-1) \le d(h) + d^-/H$ and $d(h+1) \le d(h) + d^+/H$ are required. These inequalities follow from subadditivity of d and from $d(1) \le d^+/H$, $d(H-1) \le d^-/H$ since then $d(h-1) \le d(h) + d(H-1) \le d(h) + d^-/H$. Then ρ_1 will be subadditive with slope from above at most d^+ and slope from below at least $-d^-$. To show ρ_1 subadditive, if $\rho_1(x)$ and $\rho_1(y)$ are given by $\max\{d(L(x)), d(R(x))\}$ and

$\max\{d(L(y)), d(R(y))\}$ respectively, then the previous proof still holds. Suppose either $\rho_1(x)$ or $\rho_1(y)$ is given by a term such as $d(L(x)) + d^+(x-L(x))$. In this case, x can be decreased to $L(x)$, decreasing $x+y$. But $\rho_1(x)$ decreases by more than or the same as $\rho_1(x+y)$ because nowhere is the slope steeper than d^+. Thus, it suffices to show that

$$d(L(x)) + \rho_1(y) \geq \rho_1(L(x) + y) .$$

If $\rho_1(y)$ is given by $d(L(y)) + d^+(y-L(y))$ or $d(R(y)) + d^-(R(y)-y)$, then y can be similarly moved to either $L(y)$ or $R(y)$, and subadditivity of ρ_1 follows from subadditivity of d. If $\rho_1(y)$ is given by $\max\{d(L(y)), d(R(y))\}$, then the result follows from subadditivity of d and from

$$\rho_1(L(x)+y) \leq \max\{d(L(x) + d(L(y)) ,$$

$$d(L(x)) + d(R(y))\} .$$

The simplified method given when no s^+ nor s^- is present can be easily modified by letting $d^+ = c^+$, $d^- = c^-$ and replacing γ_1 by $\min\{\gamma_1, d^+/H\}$ and γ_{H-1} by $\min\{\gamma_{H-1}, d^-/H\}$. Once this is done, there is some possibility of increasing some of the γ_h's. The idea is to first extend $\gamma_h, h = 0, \ldots, H$ to a function ρ_1 of this form. As long as $\rho_1(f_j) \leq c_j$, the resulting $\rho(f_0)$ will be a bound. Thus, the γ_h can be increased as long as the resulting ρ_1 satisfied $\rho_1(f_j) \leq c_j$. Then, reduction of γ_h to subadditive d_h results in a reduction of ρ_1 to a subadditive function and, thus, a valid inequality.

There are two other methods of extending subadditive d_h, $h = 0, \ldots, H$, to a subadditive function. These extensions were discussed in Section 6 but will be restated here. Both extensions give rise to rounding procedures along the lines given here. One extension is simply linear interpolation:

$$\rho_2(x) = \lambda d(L(x)) + (1-\lambda)d(R(x))$$

where $\lambda = (R(x)-x)/H$. Here, $d(1) \leq c^+/H$ and $d(H-1) \leq c^-/H$ are also needed when s^+ and s^- are present.

The other extension is a two-slope extension:

$$\rho_3(x) = \min\{d(L(x)) + d^+(x-L(x))),$$
$$d(R(x)) + d^-(R(x)-x)\} .$$

When s^+ and s^- are present, $d^+ \leq c^+$ and $d^- \leq c^-$ are required. Otherwise, d^+ and d^- are arbitrary positive numbers.

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