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A linear Ricardo model with varying parameters

RALPH E. GOMORY†‡ AND WILLIAM J. BAUMOL§

†Alfred P. Sloan Foundation, 630 Fifth Avenue, New York, NY 10111; and C. V. Starr Center for Applied Economics, New York University, 269 Mercer Street, Room 307, New York, NY 10003

Contributed by Ralph E. Gomory, October 24, 1994

ABSTRACT We show that linear Ricardo models with varying parameters have many of the properties usually associated with models having economies of scale.

Section 1. Introduction

This paper analyzes the classical linear Ricardo model of international trade. We will allow the efficiency parameters of the model to vary and consider all possible outcomes. We will first show that the equilibrium outcomes are confined in a region with a definite shape that has significant economic consequences. Then we will analyze motion in this region as efficiency parameters change through "learning by doing." We will show that many of the outcomes obtained in the economies of scale work of Gomory (1) and Gomory and Baumol (2), such as the conflict in the interests of two trading countries, occur in this linear model also.

We use production functions of form $e_{i,j}l_{i,j}$ and fix the sizes L_j and the demands $d_{i,j}$ of the two countries as well as n, the number of industries, so that each model is specified by $\varepsilon = \{e_{i,j}\}$. However, instead of dealing with the equilibrium point of one model we will discuss the equilibrium outcomes of all possible models—i.e., all possible ε. We will use the notation and diagrams of our earlier papers. We represent an equilibrium point with national incomes Z_1 and Z_2 and utilities U_1 and U_2 by a point p_1 in a (Z_1, U_1) diagram or a point p_2 in a (Z_2, U_2) diagram, where $Z_j = Y_j/(d_{i,1}Y_1 + d_{i,2}Y_2)$ is the relative national income of country j. We will use Cobb–Douglas utility throughout.

Equilibrium. The equilibrium conditions will be as before so that we have

$$\sum_i x_{i,1}(d_{i,1}Z_1 + d_{i,2}Z_2) = Z_1$$

$$\text{and} \sum_i x_{i,2}(d_{i,1}Z_1 + d_{i,2}Z_2) = Z_2, \qquad [1.1]$$

where the market share variables $x_{i,j}$ represent the share of the world market for the ith good that is supplied by country j. We also have the condition that at equilibrium

$$\text{if } x_{i,1} > 0, \text{ then } \frac{e_{i,1}}{w_1} \geq \frac{e_{i,2}}{w_2} \text{ and if } x_{i,2} > 0, \text{ then } \frac{e_{i,2}}{w_2} \geq \frac{e_{i,1}}{w_1}. \qquad [1.2]$$

This implies that

$$\text{if } x_{i,1} > 0 \text{ and } x_{i,2} > 0, \text{ then } \frac{e_{i,2}}{w_2} = \frac{e_{i,1}}{w_1}.$$

Normalization. Although we will consider many different $e_{i,j}$ values, we will assume that there is some best possible technology $e_{i,j}^{\max}$. Without any loss of generality we can assume that $e_{i,j}^{\max} = 1$, so $e_{i,j} \leq 1$.

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Section 2. The Region of Equilibria

We will adopt the notation (x, Z, ε) to refer to the equilibrium point whose market share variables are $x_{i,j}$, whose relative income or share of world income is $Z = (Z_1, Z_2)$, and whose efficiencies are $\varepsilon = \{e_{i,j}\}$. If we work out the utility of (x, Z, ε) we obtain points p_1 and p_2 in the (Z_1, U_1) and (Z_2, U_2) planes. Since $Z_1 + Z_2 = 1$ we can as before combine these two diagrams into a single diagram (see, for example, Fig. 1), one where Z_1 is measured from 0 to 1 and $Z_2 = 1 - Z_1$ is measured from 1 to 0, while U_1 is measured on the right-hand vertical axis and U_2 is measured on the left-hand vertical axis. In this diagram each equilibrium is represented by the two points $p_1 = (Z_1, U_1)$ and $p_2 = (Z_2, U_2)$. We will refer to p_1 and p_2 as equilibrium points for country 1 and country 2, respectively, but by this we mean no more than that there is an equilibrium (x, Z, ε) from which they are obtained. Since one point in the (U, Z) plane can correspond to many equilibria (x, Z, ε), we need the following definition:

Definition: Two equilibria (x, Z, ε) and (x', Z', ε') are called equivalent if $x = x'$, $Z = Z'$, and $x_{i,j} > 0$ implies $e_{i,j} = e'_{i,j}$.

Equivalent equilibria differ only in the efficiency of nonproducing industries, so they have the same utility and therefore correspond to the same point in the (Z, U) plane.

LEMMA 2.1. *There are equilibria for every* Z.

LEMMA 2.2. If (x, Z, ε) *is an equilibrium, so is* $(x, Z, \lambda\varepsilon)$ *for any* $0 < \lambda < 1$.

LEMMA 2.3. If (x, Z', ε) *is an equilibrium with utility* U_1, *all the points below* (Z'_1, U_1) *on the vertical line* $Z_1 = Z'_1$ *are also equilibria.*

Next we have

THEOREM 2.1. *There is a curve* $U_1 = B_1^*(Z)$ *such that every point of the* (Z, U_1) *diagram under* $U_1 = B_1^*(Z)$ *is an equilibrium point.*

Linearized Utility. Just as in refs. 1 and 2 we define the linearized utility by

$$Lu_1(x, Z, \varepsilon) = \sum_i \{x_{i,1}d_{i,1}\ln F_{i,1}(Z)q_{i,1}(1, Z_1, \varepsilon)$$

$$+ x_{i,2}d_{i,1}\ln F_{i,1}(Z)q_{i,2}(1, Z_2, \varepsilon)\}. \qquad [2.1]$$

This leads to the useful

THEOREM 2.2. *The utility* $u_1(x, Z, \varepsilon)$ *and the linearized utility* $Lu_1(x, Z, \varepsilon)$ *are equal at every equilibrium point.*

Linear Programming. We write ε_1 for vector $e_{i,j} = 1$. Consider the linear programming problem

$$\ln B_1(Z) = \text{Max}_x\, Lu_1(x, Z, \varepsilon_1)$$

$$\text{subject to} \sum_i \{d_{i,1}Z_1 + d_{i,2}Z_2\}x_{i,1} = Z_1$$

$$\text{and } 0 \leq x_{i,1} \leq 1. \qquad [2.2]$$

The solution $x(Z)$ to this problem for each Z gives us a curve $U_1 = B_1(Z)$ in the Z, U diagram. We assert that $B_1(Z)$ is always

‡To whom reprint requests should be addressed.

above the region of equilibria. More precisely (recalling *Theorem 2.1*),

THEOREM 2.3. $B_1^*(Z) \leq B_1(Z)$.

In addition the two curves can not be very far apart. Specifically:

THEOREM 2.4. $1 \leq B_1(Z)/B_1^*(Z) \leq (1/min(w_1/w_2, w_2/w_1))^D$, *where* $w_1 = Z_1/L_1$ *and* $w_2 = Z_2/L_2$, *and* $D = max_i\ d_{i,1}$.

From this follows easily:

THEOREM 2.5 (CONVERGENCE THEOREM). *For any* Z, $B_1^{n*}(Z) \rightarrow B_1^n(Z)$ *as* $n \rightarrow \infty$ *provided that* $D^n \rightarrow 0$.

Therefore for problems with large numbers of industries the space of all possible points corresponding to equilibria becomes almost identical with the space under the curve $B_1(Z)$. The curves $B_1(Z)$ and the corresponding curve $B_2(Z)$ for country 2 are the dark lines in Fig. 1, and the actual upper boundaries $B_1^*(Z)$ and $B_2^*(Z)$ lie between the dark and light lines. The example used in Fig. 1 was a 22-industry example.

The reasoning of Gomory and Baumol (2) can be applied to show that the shape of the regions of outcomes of the two countries is always as in Fig. 1. $B_1(Z)$ rises steadily as Z_1 increases, then turns down to a level which represents the utility country 1 would obtain in autarky with all $e_{i,1} = 1$. The peak for country 1 is always to the right of the level at which wages are equal in the two countries, and the peak for country 2 is always to the left of that point. The equal wage point is marked by the vertical line on the upper edge of the box in Fig. 1.

This regional shape already has important consequences. It shows that the best result for country 1 occurs when it has a large share of world income and when country 2 has relatively little. Similarly, country 2 does best when it has most of the world income, which means that it produces most of the world's goods, and country 1 has relatively little. As we will see later, if country 2 improves its technology, the effect is to give it a larger share of world production and world income, thus moving country 1 leftward and downward from its peak. The diagram also shows that it is possible to have too large a share of world income. Nevertheless, there is inherent conflict in the interests of the two nations; what is best for one tends to be bad for the other.

While this result is independent of any concept of motion in this outcome space it is nevertheless illuminating to consider such motions.

Section 3. Natural Motion in Outcome Space

We will consider the motion of the equilibrium outcome as the parameter ε varies with time. Any changes in the parameters

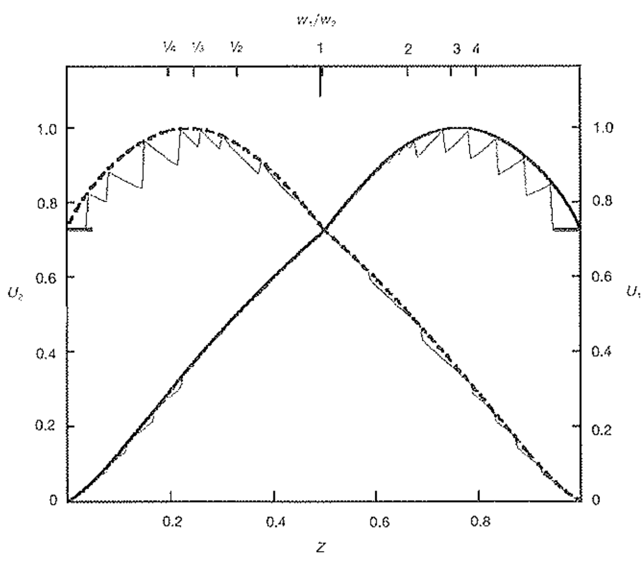

FIG. 1. Income share vs. utility regions for the two countries.

ε will produce a series of new equilibria (x, Z, ε). Each equilibrium has a utility, so the result is motion of the corresponding point p in (Z, U) space.

Rules Governing Motion. The rules for change in the efficiency parameters to produce *natural motion* will be based on the observation that a producing country becomes more efficient through practice and acquisition of knowledge, unless its efficiency is already maximal. We will also often assume that the efficiency of a nonproducing country tends to decay.

In the next section we will also consider motion that goes beyond these rules, as when a nonproducing country targets an industry and improves its efficiency as part of a deliberate entry effort. However, in this section we will consider only the simplest case that we call natural motion.

Assumptions About the Limits of Efficiency. Motion will also be limited by some further assumptions. These can take several forms. (*i*) We could assume that if both countries produce each can attain maximum efficiency. (*ii*) We could assume that the countries have some natural endowments that permit some maximum natural efficiency $e_{i,j}^*$ so that efficiency has an upper limit that is peculiar to the country, $0 \leq e_{i,j} \leq e_{i,j}^*$. We will adopt assumption (*ii*) at this point and treat (*i*) as a special case.

Natural Motion Rules. We will assume that natural motion is governed by the following rules: (NM1) A producing industry ($x_{i,j} > 0$) gains efficiency. If it remains a producing industry its efficiency will eventually approach its maximum, $e_{i,j}^*$ (NM2) A nonproducing industry ($x_{i,j} = 0$) loses efficiency. If it remains a nonproducing industry its efficiency will eventually approach 0. (NM3) Country 1 does not lose comparative advantage in the industries in which it is a producer compared with the industries in which it is a nonproducer. More formally: If $x_{i,1} > 0$, and $x_{k,1} = 0$, then $(e_{i,1}/e_{i,2})/(e_{k,1}/e_{k,2})$ is nondecreasing with time.

Natural Motion: Fixed Points. We can define a fixed point of natural motion as an equilibrium point (x, Z, ε) where x, Z and the corresponding U do not change. Therefore the corresponding $p(t) = (Z, U)$ is fixed in the (Z, U) plane.

LEMMA 3.1. *A necessary condition for a point* (x, Z, ε) *to be a fixed point is that for every* $x_{i,j} > 0$, $e_{i,j} = e_{i,j}^*$.

This leads us to the following theorem, in which we use ε^* for the vector $\{e_{i,j}^*\}$:

THEOREM 3.1 (FIXED POINTS). *Every point of* (Z, U) *corresponding to a fixed point of natural motion lies in the region bounded above by* $B_1(Z, \varepsilon^*)$ *and below by* $BL_1(Z, \varepsilon^*)$, *where* $B_1(Z, \varepsilon^*)$ *is defined by the linear program*

$$ln\ B_1(Z, \varepsilon^*) = Max_x\ Lu_1(x, Z, \varepsilon^*)$$

$$subject\ to\ \sum_i \{d_{i,1}Z_1 + d_{i,2}Z_2\}x_{i,1} = Z_1\ and\ 0 \leq x_{i,1} \leq 1 \qquad [3.1]$$

and $BL_1(Z, \varepsilon^*)$ *is defined by the corresponding minimization problem*

$$ln\ BL_1(Z, \varepsilon^*) = Min_x\ Lu_1(x, Z, \varepsilon^*)$$

$$subject\ to\ \sum_i \{d_{i,1}Z_1 + d_{i,2}Z_2\}x_{i,1} = Z_1\ and\ 0 \leq x_{i,1} \leq 1. \qquad [3.2]$$

Now we have shown that all fixed points lie in the familiar region between the upper and lower boundaries of a linear program with the linearized utility objective function. The shape of such a region is known from our earlier work and is illustrated in Fig. 2. If we look at this in a little greater detail we can easily see that

THEOREM 3.2 (SPECIALIZED EQUILIBRIA). *Every specialized solution* (x, Z) *of the linear program* **3.1** *corresponds to a fixed*

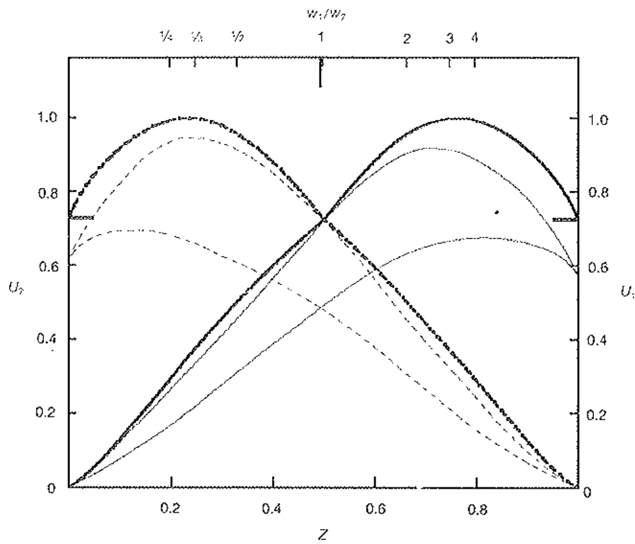

FIG. 2. Regions containing the fixed points of natural motion. B_1 and BL_1 are the light continuous lines and B_2 and BL_2 are the light dashed lines.

point of natural motion. These points lie between the curves B_1 *and* BL_1.

THEOREM 3.3 (NONSPECIALIZED EQUILIBRIA). *All nonspecialized equilibria that are fixed points lie on the* n *vertical lines* $w_1/w_2 = e^*_{i,1}/e^*_{i,2}$ *and within the region between the boundary curves.*

In *Theorems 3.1, 3.2,* and *3.3* we have used only the properties NM1 and NM2 of natural motion. If we now invoke NM3 we will be able to show that any starting point p has a natural motion which approaches one of the fixed points of the region. We obtain the following result.

THEOREM 3.4 (CONVERGENCE). *Under natural motion all points* p *approach points in the region of fixed points.*

The nature of this convergence is to "freeze in" something close to the initial shares of world income. Income itself increases to higher levels that result from the improved efficiencies. This freezing in probably requires NM3; in its absence more complicated natural motion possibilities may arise. However, aside from natural motion there are other possibilities.

Section 4. Less Natural Motion—Capture of Industries

Next we will look at motion that does not occur "naturally," the capture of an industry as the result of effort by one country to increase its $e_{i,j}$ in an industry in which it is a nonparticipant. We will consider here only the simplest case, in which the starting point is a specialized equilibrium. However, the same results can be obtained with more complexity if the starting point is nonspecialized.

Preparing for Capture. In industry k in which country 1 is the nonproducer, country 1 increases its $e_{k,1}$ until it obtains parity with country 2—i.e., $e_{k,1}/w_1 = e_{k,2}/w_2$. This is possible only if $(w_1/w_2)e_{k,2} \leq 1$, so this is more likely to be a possibility for the lower wage country. At the new level of $e_{k,1}$ we have a new equilibrium point. This new equilibrium point is equivalent to the starting equilibrium point, so we have the same point in the Z–U plane.

Shift of an Industry. There are many possible versions of the capture process; we will discuss what may be the simplest. The efficiency of country 2 in the kth industry, $e_{k,2}$,¶ remains fixed as production shifts from country 2 to country 1. County 1 increases its efficiency just enough to maintain parity in the face of its increasing wage. In model terms we vary the parameters in the linear model by requiring $e_{k,1}(Z_1)/w_1 = e_{k,2}/w_2$ as country 1 increases its income Z_1 and wage w_1. As production shifts and Z_1 increases we get a curve of new equilibrium points moving to the right from the original equilibrium. We need to know whether it is going up or down for country 1 and up or down for country 2. The result can be summarized as follows:

Effects of Capture. As the industry shifts, the capture process either increases consumption or leaves things the same in every good for country 1. The effect on country 2 is either no change or a decrease. *So the effect of the capture process is to increase country 1's utility and decrease country 2's utility.*

The point corresponding to this equilibrium moves rightward. It moves up for country 1 and can move either up or down for country 2, depending on whether the depressing effect of capture is or is not outweighed by learning in all other industries during the capture process. This continues until the entire output comes from country 1 and the whole industry has been captured.

So we have once again, and without benefit of explicit economies of scale, the conflict that goes with *share* and retainable industries. What makes these industries retainable is the learning-by-doing aspect of natural motion.

¶Because k may be unable to match the other countries' effort or it may already be at $e_{i,j}$.

1. Gomory, R. E. (1994) *J. Econ. Theory* **62**, 394–419.
2. Gomory, R. E. & Baumol, W. J. (1994) *Shares of World Output, Economies of Scale, and Regions Filled with Equilibria* C. V. Starr Economic Res. Rep. 9429 (New York Univ., New York).