Math. Program., Ser. B 96: 321–339 (2003)

Digital Object Identifier (DOI) 10.1007/s10107-003-0388-4

Ralph E. Gomory · Ellis L. Johnson · Lisa Evans

Corner Polyhedra and their connection with cutting planes

Received: May 24, 2001 / Accepted: August 2002
Published online: March 21, 2003 –

Abstract. We review the necessary background on Corner Polyhedra and use this to show how knowledge about Corner Polyhedra and subadditive functions translates into a great variety of cutting planes for general integer programming problems. Experiments are described that indicate the dominance of a relatively small number of the facets of Corner Polyhedra. This has implications for their value as cutting planes.

Introduction

It is the purpose of this paper to show how facets of the Corner Polyhedra, the convex hull of integer solutions to Gomory's group problem [1], can be obtained by empirical investigations. The "shooting experiment," presented in section 1, quickly generates many facets and gives an intuition about which facets are "important." We then show how these "important" facets can be used to construct a great variety of cutting planes for use in practical integer programming.

This paper is divided into three parts:

Part 1 Corner Polyhedra, shooting theorem, and facets;
Part 2 Application of Corner Polyhedra to generating cutting planes;
Part 3 General remarks.

This paper makes use of results from three earlier papers [1–3]. The first introduced Corner Polyhedra and contains many of the fundamental theorems. The last two extended the understanding of Corner Polyhedra in significant ways, developed a theory for mixed integer programs, and developed a simpler framework for connecting Corner Polyhedra and general cutting planes. This paper is self contained in the sense that it contains and explains the theorems from these papers when they are needed, although it does not give their proofs.

R.E. Gomory: Sloan Foundation, New York, NY, USA

E.L. Johnson: Georgia Institute of Technology, Atlanta, Georgia, USA

L. Evans: Georgia Institute of Technology, Atlanta, Georgia, USA

Mathematics Subject Classification (1991): 20E28, 20G40, 20C20

1. Corner Polyhedra, shooting theorem, and facets

1.1. Corner Polyhedra

The Corner Polyhedra are a relaxation of the non-negativity constraint on the basic variables. In Figure 1, which shows a two dimensional Corner Polyhedron, the dark area is the integer programming polyhedron derived from a linear programming problem whose constraints are shown as lines. The large letter V marks the vertex where the non-basic constraints meet. The light gray area, which continues off the figure between the two non-basic constraint lines, is added when the basic constraints are relaxed and the combined dark and light gray areas make up the Corner Polyhedron. The Corner Polyhedron is the convex hull of the integer points in the area defined only by the non-negativity of the non-basic variables. In a Corner Polyhedron, therefore, it is required that the non-basic variables be both integer and non-negative, but the basic variables are only required to be integer.

Corner Polyhedra are connected to the original integer programming problem in two ways. (1) The facets of the Corner Polyhedron are cutting planes of the original linear programming problem. This is the aspect of Corner Polyhedra that we will emphasize in this paper. (2) The problem is identical with the integer programming problem if the right hand sides are large enough. This gives rise to the theory of asymptotic integer programming. The one-dimensional version of asymptotic integer programming appeared first in Gilmore and Gomory [4].

Corner Polyhedra are the simplest integer programs in the sense that they are present in every integer program, so if we can find out something about them it is likely to be useful. Fortunately, they turn out to be much simpler and more intelligible than the more general integer program.

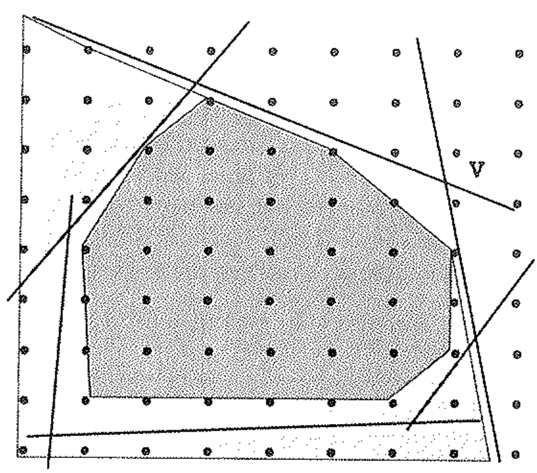

Fig. 1. The Integer Programming Problem and Corner Polyhedron

Consider an ordinary $m \times (m + n)$ linear programming problem with integer data. Consider a vertex and the associated basic feasible solution which divides the variables into m basic and n non-basic variables.

$$\begin{gathered} Max \quad cx \\ Ax = b,\ x \geq 0 \end{gathered}$$

where $A = (B, N)$ and $x = (x_B, x_N)$, so

$$Bx_B + Nx_N = b;$$

or, equivalently,

$$Ix_B + (B^{-1}N)x_N = B^{-1}b. \tag{1}$$

The Corner Polyhedron associated with this basic feasible solution is obtained by allowing the basic variables to have any sign. The set of all integer combinations of the columns of B gives a lattice L in m-space. To produce an integer solution to the Corner Polyhedron problem, the n non-basic variables times their (integer) columns must add up to the right hand side m-vector b, modulo the lattice L. An equivalent statement, obtained from equation (1) is: The non-basic columns, transformed by B^{-1}, must add up to the right hand side $Mod 1$. In the notation used below, saying two vectors u and v are equivalent $Mod 1$ means the corresponding elements u_i and v_i of the vectors are equivalent $Mod 1$ for each index i.

$$(B^{-1}N)x_N = B^{-1}bMod 1. \tag{2}$$

The factor group of all integer vectors in m-space taken mod$B$ form a finite group G. G is also obtained from reducing $Mod 1$ the (usually non-integer) vectors in m space that are B^{-1} transforms of integer vectors. Either way, G has D elements, where D is the value of the determinant of B. The group element corresponding to $B^{-1}N_i$, the transformed ith column of N, is $B^{-1}N_i$ taken $Mod 1$.

Consequently, in place of (2), we can write, using g_i for the group element corresponding to N_i and g_0 for the group element corresponding to the right hand side $B^{-1}b$:

$$\sum_{g \in G} t(g)g = g_0. \tag{3}$$

Here the variables $t(g)$, which are positive integers or 0, are just the non-basic variables x_i adjusted for possible duplication – more than one column of N may map into the same group element g.

Solutions $\{t(g)\}$ to (3) are integer points within the Corner Polyhedron. More concretely, if we find a solution $\{t(g)\}$, we can translate it back to the non-basic variables, possibly in more than one way. The non-basic variables then determine the basic variables, which will necessarily then turn out to be integers, although not necessarily non-negative integers.

If we refer to the space of the variables t in (3) as T-Space, we have the picture of Figure (2). The t-variables are the non-basic variables adjusted for possible duplication.

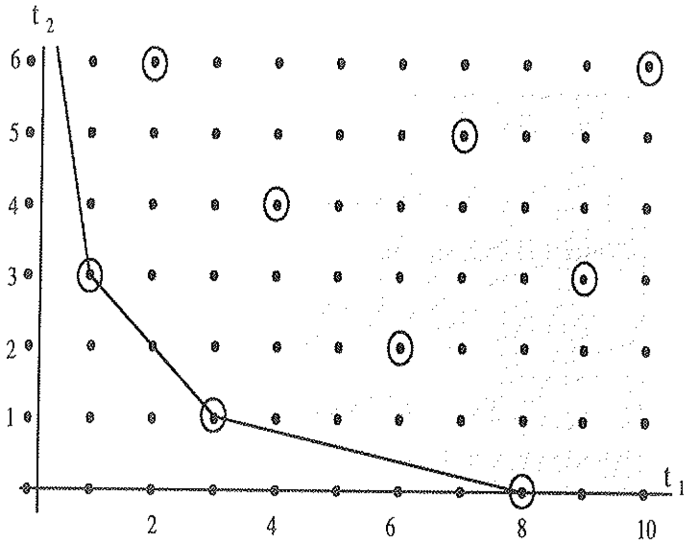

Fig. 2. T-Space

The dark dots are any integer t-values and the circled dots are the non-negative t-values that satisfy the group equation. These are the values of the non-basic variables that make the columns of N add up to the right hand side b, $Mod B$.

The convex hull of these $t(g)$ in T-space we call $P(G, N, g_0)$. The facets of these polyhedra $P(G, N, g_0)$ leave all the solutions to the Corner Polyhedron problem on one side. These solutions include all the integer solutions to the original integer programming problem. *Therefore, the facets of $P(G, N, g_0)$ are always "cutting planes" for the integer programming problem.*

However, the polyhedron $P(G, N, g_0)$ is still rather problem specific. It depends on which group elements are actually represented by non-basic columns $B^{-1}N_i$ and which are not. We can move to a more problem independent approach by introducing Master Polyhedra.

1.2. Master Polyhedra

The Master Polyhedron is the polyhedron $P(G, N, g_0)$ obtained whenever N contains a column $B^{-1}N_i$ for every group element $g \in G$. We use the notation $P(G, g_0)$ for the Master Polyhedron with right hand side element g_0. Master Polyhedra have many useful properties.

Master Polyhedra contain all the facets of the other polyhedra, the ones in which some group elements are missing. This is expressed in the Master Polyhedron Theorem, which appears as Theorem 12 in Gomory [1] and is restated below. The theorem says that a particular polyhedron $P(G, N, g_0)$ can be obtained from the Master Polyhedron by intersecting the Master Polyhedron with the subspace of the t-variables that are represented by the columns of N.

Theorem 1 (Master Polyhedron Theorem).

$$P(G, N, g_0) = P(G, g_0) \cap E(N),$$

Here $E(N)$ is the subspace of variables $t(g)$ for which there is a $B^{-1}N_i$ corresponding to g.

This means that the facets of the Master Polyhedron $P(G, g_0)$ are cutting planes for *any* integer programming problem with Corner Polyhedron $P(G, N, g_0)$ for any N. Note that it is not asserted that each facet of the Master Polyhedron $P(G, g_0)$, restricted to the subspace $E(N)$, is a facet of $P(G, N, g_0)$. Rather, it is asserted that the inequalities obtained by restricting the facets of $P(G, g_0)$ to $E(N)$ *all* leave $P(G, N, g_0)$ to one side – therefore they are *all* cutting planes – and that *among* these inequalities are all the facets of $P(G, N, g_0)$.

The next important point is that the facets of the Master Polyhedron, including therefore all the facets of the non-master polyhedra, can be obtained as the basic feasible solutions of a very structured ordinary *linear* programming problem.

To see this we need to introduce π-space. π-space, like T-space, is a linear space with one dimension for each group element of G. A point in π-space is a vector (π, π_0), where $\pi \in R^{|G^+|}$ and $\pi_0 \in R$, that gives a real value for each group element $g \in G^+$, where G^+ is G without the zero element $\underline{0}$ of G. In our notation below, $\pi(g)$ refers to the component of the vector π that corresponds to the element $g \in G^+$.

There is a very close connection between the vertices of a very structured polyhedron $\Pi(G, g_0)$ in π-space, and the facets of the integer polyhedron $P(G, g_0)$. We obtain $\Pi(G, g_0)$ by first writing down all the conditions for π to be subadditive on the group elements, i.e. that we always have $\pi(g) + \pi(g') \geq \pi(g + g')$ for any $g, g' \in G$. These inequalities are really the underlying conditions in defining $\Pi(G, g_0)$ in (4) below. The equality conditions $\pi(g) + \pi(g_0 - g) = \pi(g_0)$ in (4) are derived from the inequality condition by some additional reasoning that shows that we can require this particular inequality to always be an equality for basic solutions. The other conditions in (4) are normalizations. The close connection of this structured π-space polyhedron with the T-space master polyhedron $P(G, g_0)$ is given by the following theorem, which also appears as Theorem 18 in [1], except that here we do not discuss the case $g_0 = \underline{0}$:

Theorem 2 (Facets of the Master Polyhedra). *The inequality*

$$\sum_{g \in G^+} \pi(g) t(g) \geq \pi_0$$

given by the vector (π, π_0) with non-negative $\pi(g)$ and with $\pi_0 > 0$, is a facet of the polyhedron $P(G, g_0)$, $g_0 \neq \underline{0}$, in T-space if and only if it is a basic feasible solution to the system of equations and inequalities in π-space:

$$\begin{aligned} \pi(g_0) &= \pi_0 \\ \pi(g) + \pi(g_0 - g) &= \pi_0 \quad g \in G^+, g \neq g_0 \\ \pi(g) + \pi(g') &\geq \pi(g + g') \quad g, g' \in G^+ \end{aligned} \tag{4}$$

The basic feasible (π, π_0) of the theorem give us *facets of the Master Polyhedron in T-space*. This means that we have $\pi \cdot t \geq \pi_0$ for all $t \in P(G, g_0)$ and that $\pi \cdot t = \pi_0$ is a facet of $P(G, g_0)$. However in π-space, the π are *vertices* of the polyhedron $\Pi(G, g_0)$ defined by the inequalities and equalities of [1]. This theorem enables us to compute facets of the integer polyhedron $P(G, g_0)$ by finding basic feasible solutions of the ordinary linear set of equations and inequalities defining $\Pi(G, g_0)$.

The Master Polyhedra can be considered to be the irreducible atoms of integer programming. Because of (4), *small* master polyhedra can actually be computed and all their facets and vertices found. Because of the Master Polyhedron Theorem the *facets of these polyhedra are cutting planes* for any integer programming problem having that group as the group of its Corner Polyhedron. Table 1 shows 3 of the 12 facets of $P(G_{10}, 9)$, with G_{10} being the cyclic group of order 10, and 9 denoting the 9th group element. Table 2 also shows 3 of the at least 782 facets of $P(G_{20}, 19)$ based on the cyclic group of 20 elements and right hand side 19. Gomory [1] tabulates the facets and vertices for all groups of order 11 or less.

1.3. The problem of size

However, the size of group G for which we can reasonably expect to compute the entire Master Polyhedron $P(G, g_0)$ is very small compared to the size of the groups we would encounter in practical problems. These groups can easily contain millions or billions of elements. This might seem to preclude the use of groups for practical use, for example in generating cutting planes; but this is definitely *not* the case. There are several approaches which let us make use of our knowledge of small Master Polyhedra to generate cutting planes for arbitrarily large problems. The earliest such approach was in Gomory [1], which made use of lifting to move facets of small polyhedra into large ones, and made use of automorphisms to translate a single facet into many different facets. Group characters were also used to map the group G into a cyclic group and avoid needing to know the structure of G.

Later in this paper we will describe and use the method used in Reference [3], which generates cutting planes without even knowing what group G is involved, and which extends to mixed integer as well as all-integer problems.

Table 1. First three facets of $P(G_{10}, 9)$

π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_0
1	0	1	0	1	0	1	0	1	1
1	2	3	4	0	1	2	3	4	4
4	3	2	6	0	4	3	2	6	6

Table 2. First three facets of $P(G_{20}, 19)$

π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}	π_{16}	π_{17}	π_{18}	π_{19}	π_0
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1
1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3	3
1	2	3	4	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4	4

1.4. Knowledge about Master Polyhedra

All these approaches are helped by knowledge of the Corner Polyhedra. In Gomory and Johnson [2], it was shown that the number of facets of Master Polyhedra grows exponentially. However, knowing that there are a vast number of facets leaves open the question of whether there are many facets of roughly equal importance, or whether there are relatively few major facets and thousands of tiny ones. Probably for practical problems, we would prefer to deal with the major facets if they form a significant portion of the polyhedron. How can we shed some light on the question of there being major and minor facets, and how can we find the more significant ones?

1.5. Shooting theorem

Conceptually, this problem could be approached as follows: Shoot arrows (more precisely, random directions from the origin) at the facets and see which they hit. Many random directions hitting the same facet would suggest that it is a big facet, few or none would suggest it was small. Perhaps we could then see what the distribution of facet sizes is, and which facets turn out to be the big ones. Shooting experiments were first used by Kuhn [5] to measure facets of the traveling salesman polytope, and required a list of all extreme points of the polytope *a priori*. Kuhn shot random vectors from the approximate center of the bounded traveling salesman polytope to see which facet was hit first.

If we accept this approach, there is still the question of how to do this shooting. If the facets of the Master Polyhedron were known, we could proceed as follows: Choose a random direction v in the first quadrant, increase the vector v by multiplying by a scalar λ, test the vector $v\lambda$ against the various facets (π^i, π_0) to see which side of each facet it is on, or, equivalently, which facet defining inequalities are satisfied by $v\lambda$. Keep increasing λ until $v\lambda$ gets to be on the far side, the Master Polyhedron side, of all those facets. When $v\lambda$ gets beyond *all* the facets, it is *in* the polyhedron, so the *last* facet it gets beyond is the sought-after facet containing the point where $v\lambda$ hits the polyhedron. Put another way, if $v\lambda$ lies beyond all the facets but one, and lies on that one, that facet is the one hit by the random direction v. (Since the facets of the Master Polyhedra have nonnegative coefficients, this hit will occur for large enough λ unless the random v has 0's in it in positions that include all the positive elements of one of the facets of $P(G, g_0)$. The probability of randomly generating such a v is 0).

This conceptual process requires having the facets to test against the random direction $v\lambda$. Except for the smallest Master Polyhedra, we do not have those facets. Remarkably, there is a way of doing shooting experiments without knowledge of the facets. This is given in the following theorem, which was introduced by Ralph Gomory at a lecture at Georgia Tech in 1998 that also included some very preliminary computational results:

Theorem 3 (Shooting Theorem). *The facet of $P(G, g_0)$ hit by the random direction v is the facet given by the minimizing basic feasible solution of the objective function v in π-space subject to the constraints of $\Pi(G, g_0)$.*

In other words, the facet hit by v is obtained by solving the linear programming problem:

$$\begin{aligned} \min \quad & v\pi \\ \text{subject to} \quad & \\ \pi(g_0) &= \pi_0 \\ \pi(g) + \pi(g_0 - g) &= \pi_0 \quad g \in G^+, g \neq g_0 \\ \pi(g) + \pi(g') &\geq \pi(g + g') \quad g, g' \in G^+. \end{aligned} \tag{5}$$

Proof. If π is the minimizing solution to (5), choose a positive scalar λ so that $\pi v\lambda = \pi_0$. Such a λ exists because v is a vector from the origin into the non-negative orthant, so some positive multiple of it must intersect with the inequality $\pi t \geq \pi_0$. Since π minimizes v in (5) we have $\pi^i v \geq \pi v$ for all basic feasible solutions π^i to (4). This implies $\pi^i v\lambda \geq \pi v\lambda = \pi_0$. From the Master Polyhedron Theorem we know that the facets of $P(G, g_0)$ are the (π^i, π_0). Therefore the relations $\pi^i v\lambda \geq \pi v\lambda = \pi_0$ imply that $v\lambda$ lies on the Master Polyhedron side of all the other facets π^i of $P(G, g_0)$ and that $v\lambda$ lies on the facet (π, π_0). Therefore π is the facet hit by the random direction v. This ends the proof.

This theorem asserts that one linear programming calculation produces the facet hit by one random direction. The theorem gives us a way of discovering facets of any polyhedron $P(G, g_0)$ for which the linear programming problem (5) is solvable. We will also tend to discover large facets first, as they then to be the ones that are hit. Note also that the linear programming problem is highly structured, and that the rows, which are much more numerous than the columns, could be produced when needed by row generation methods and a dual simplex approach.

Making use of this theorem, Johnson and Evans wrote and ran a shooting program capable of dealing rapidly with problems (5). They were able to fire off 10,000 shots at each polyhedron that they investigated. They have obtained data on polyhedra up to group size 30.

1.6. Shooting results: the concentration of hits

Table 3 contains data about the number of facets receiving hits in the shooting experiment, and suggests that the concentration of hits is on a small percentage of the facets. The first column gives the name of the master polyhedron corresponding to the shooting experiment. The second column gives the number of different facets hit in 10,000 shots. In the third column we take the facets that were hit most, and show how many of these it took to absorb 50% of the hits. These experiments indicate a strong concentration of hits on relatively few facets. Conceptually we have to consider that these results could result from the angle of facets rather than from size. There could be many big faces, but they could be tilted away from the origin so they are hard to hit. We discuss this possibility in Appendix A and show that it seems very unlikely.

In Figures 3–4 we show in more detail the distribution of hits on the 30 facets that account for 50 percent of the hits on the (at least) 605 facets of $P(G_{19}, 18)$ and the 11 facets that account for 50 percent of the hits on the (at least) 781 facets of $P(G_{20}, 19)$.

Table 3. Shooting experiment results

Polyhedron	Number of facets hit in 10,000 tries	Number of facets to 50% of hits
$P(G_{18}, 2)$	151	8
$P(G_{18}, 3)$	479	11
$P(G_{18}, 6)$	207	17
$P(G_{18}, 9)$	505	12
$P(G_{18}, 17)$	309	7
$P(G_{19}, 18)$	605	30
$P(G_{20}, 2)$	341	10
$P(G_{20}, 4)$	587	19
$P(G_{20}, 5)$	929	20
$P(G_{20}, 10)$	371	25
$P(G_{20}, 19)$	781	11

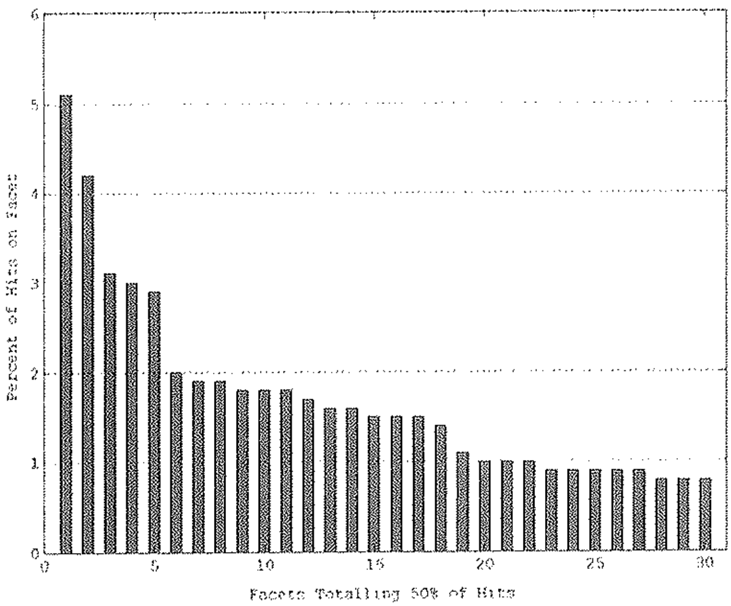

Fig. 3. $P(G_{19}, 18)$

In Figures 3 and 4 the facets are ordered according to their number of hits. The first bar gives the percent of hits on the facet with the most hits, etc. The distribution of hits is much more concentrated on a few facets for $P(G_{20}, 19)$ than for $P(G_{19}, 18)$.

Polyhedra whose groups have subgroups tend to have some very simple facets lifted up from subgroups. These simple facets get a lot of hits. Since 19 is a prime, G_{19} has no subgroups. However, G_{20} has lots of subgroups and this appears in the facets that get the most hits as we will see below. All of this reflects the fact that we are dealing with very structured Master Polyhedra.

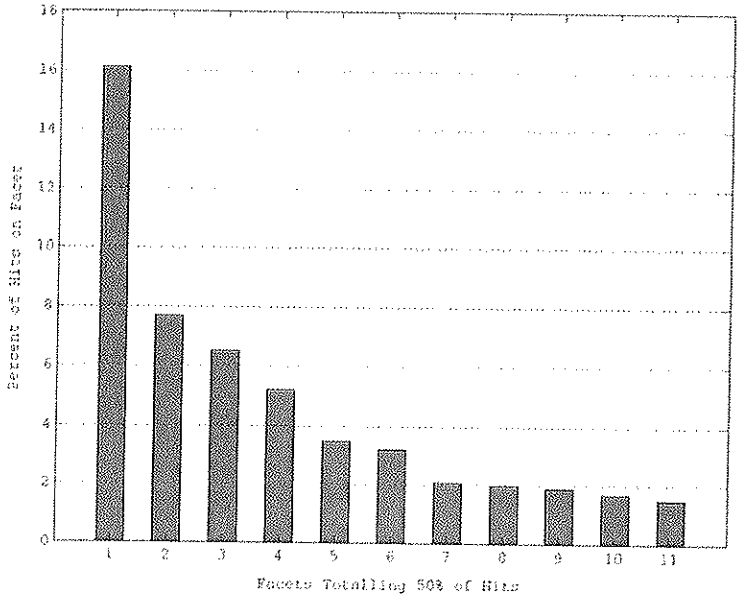

Fig. 4. $P(G_{20}, 19)$

1.7. Shooting results: dominant facets and their structure

When we look at the actual facets there is clearly lots of structure. If we are dealing with cyclic groups, which we have in all shooting experiments up to this point, we can represent the group elements as integers on the real line Mod n. Then we can plot the $\pi(g)$ value π_i for the ith group element as a dot above the ith integer point, and then connect these dots, the values of the π_i, with straight lines. The group element $\underline{0}$ is at the origin and $\pi(\underline{0})$ is always 0. The resulting diagrams for the first four most hit facets of $P(G_{15}, 14)$ are shown in Figures 5–8.

In each diagram, the facets have been normalized so that the right hand side group element has π-value = 1, or, equivalently, height = 1. The first three of these facets are repetitions of facets of smaller groups that have been "lifted up" to the larger group. Reference [1], Section D, describes various methods of lifting up facets from subgroups. The fourth facet is the Gomory mixed integer cut.

However, if we look at $P(G_{13}, 12)$, we see something quite different. Since 13 is prime, there are no subgroups. However, there are lots of automorphisms that map facets from problems with different right hand sides to facets for this problem. The two most hit facets of $P(G_{13}, 12)$ are shown in Figures 9–10. Both are quite simple. The two segment curve appearing in 9 is the Gomory mixed integer cut. In this case it has right hand side element 12. The third and fourth most hit facets are more complicated but they are both automorphisms of simpler facets with different right hand sides. The most extreme case is the facet which received the fourth most hits, which is shown in Figure 11. We assert that the curve in Figure 12 produces the curve in Figure 11 through an automorphism. We can verify from Figures 11–12 that if we refer to the first curve as

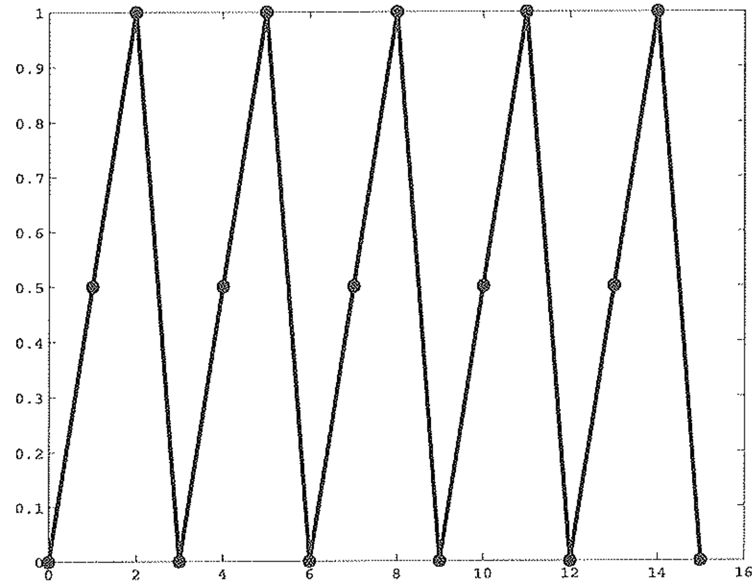

Fig. 5. Facet of $P(G_{15}, 14)$ with most hits

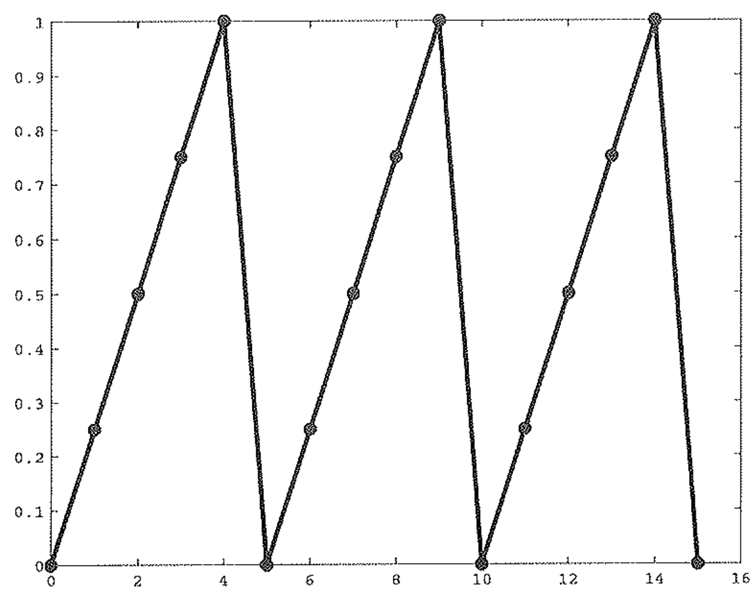

Fig. 6. Facet of $P(G_{15}, 14)$ with 2nd most hits

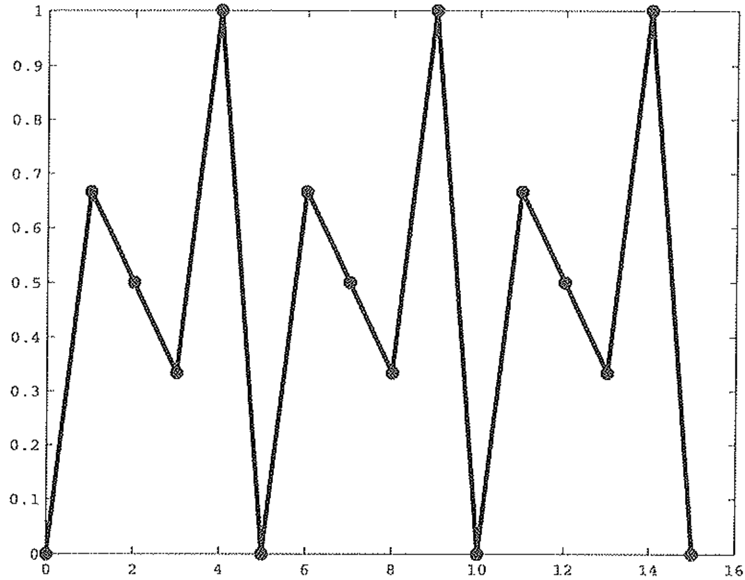

Fig. 7. Facet of $P(G_{15}, 14)$ with 3rd most hits

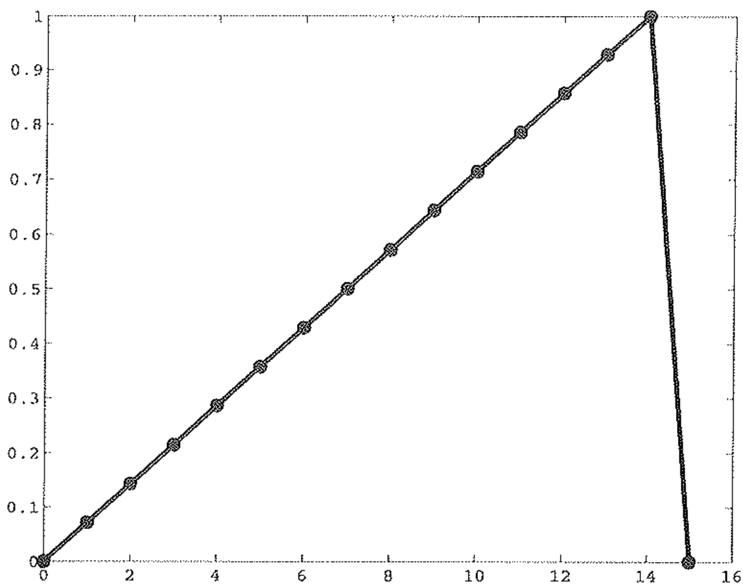

Fig. 8. Facet of $P(G_{15}, 14)$ with 4th most hits

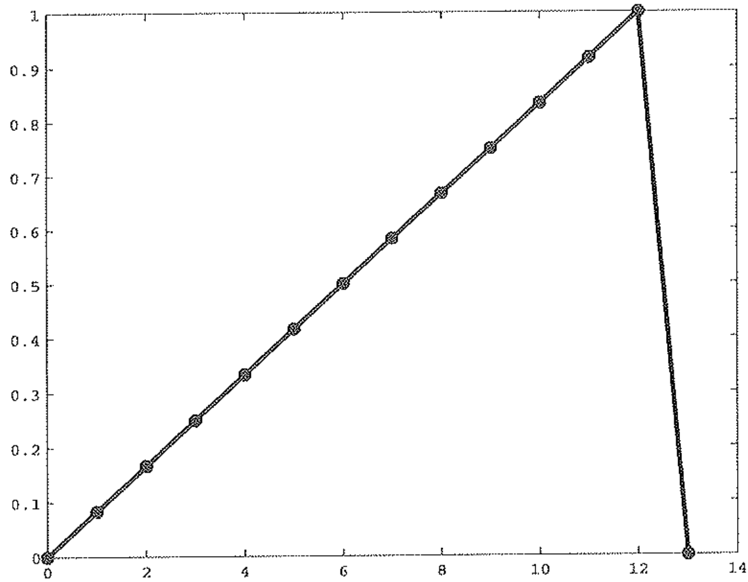

Fig. 9. Facet of $P(G_{13}, 12)$ with most hits

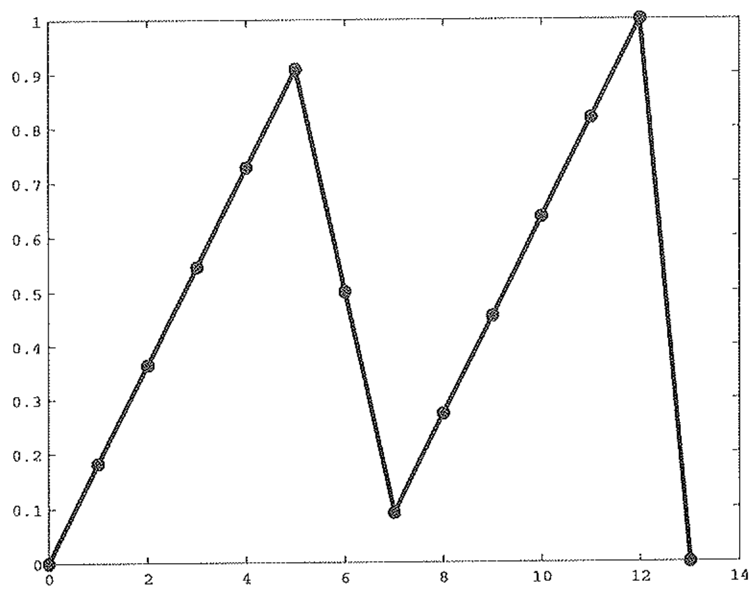

Fig. 10. Facet of $P(G_{13}, 12)$ with 2nd most hits

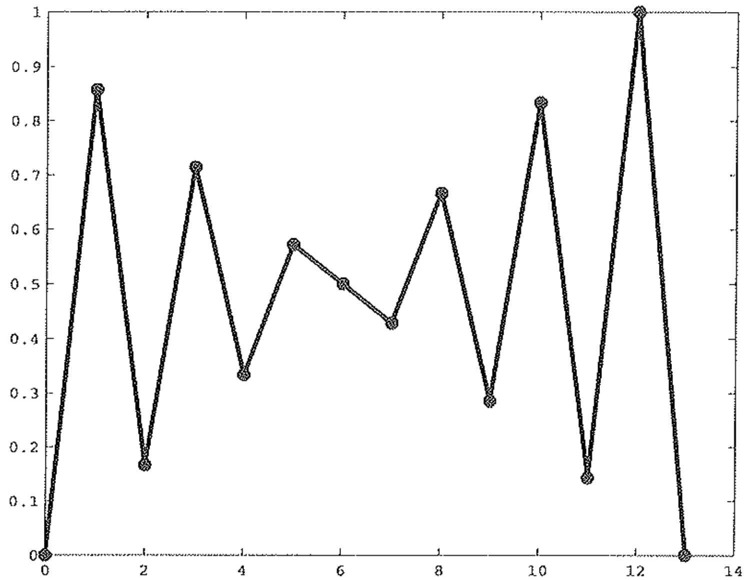

Fig. 11. $\pi(u)$ – Facet of $P(G_{13}, 12)$ with 4th most hits

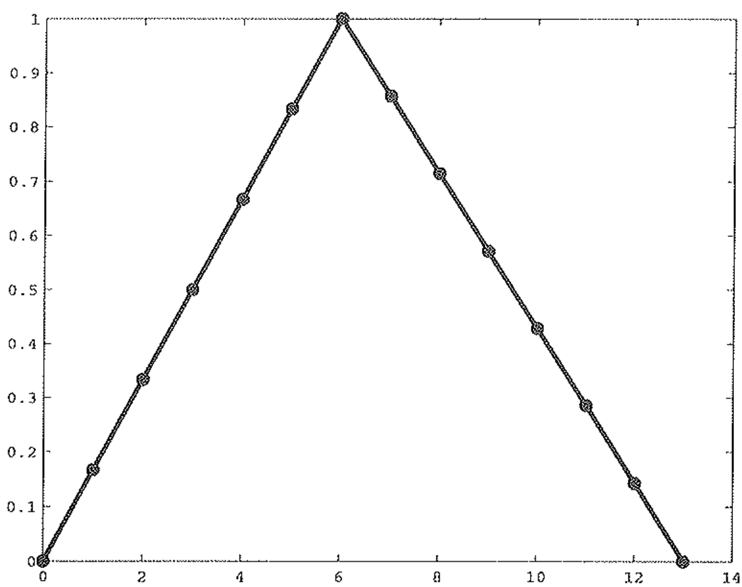

Fig. 12. $\pi^*(u)$ – The related Automorph with RHS – .6: $\pi(2u) = \pi^*(u)$

$\pi(u)$ and the second curve as $\pi^*(u)$, then the values on the group elements are related by $\pi(2u) = \pi^*(u)$.

In addition to the structure within a given Master Polyhedron, there is significant persistence of facets from one Master Polyhedron to another. Recall that Table 1 shows the first 3 facets of $P(G_{10}, 9)$, the Master Polyhedron for the cyclic group of 10 elements with right hand side 9, and Table 2 shows the first 3 facets of the much larger $P(G_{20}, 19)$. Two of the three facets follow the same pattern. While for groups having common subgroups this can be explained by facets being lifted up, there is the same tendency among the prime cyclic groups. The same two facets that are the most hit in $P(G_{13}, 12)$, shown in Figures 9–10, are also the two most hit facets in $P(G_{17}, 16)$ and $P(G_{19}, 18)$.

What these experiments tend to show is that there are a relatively small number of facets that do play a major role in the Corner Polyhedra we have been able to examine. Many of these facets have simple and recognizable structures. The structure of all these facets is examined in far greater depth in [6]. All of the facets, large or small, are of course cutting planes for integer programming problems having Corner Polyhedra based on these groups. We would have lots of promising cutting planes if we managed to limit ourselves to problems in which the group size was always 20 or less. However, we will now go on to show that the knowledge of the structure of these small groups gives us cutting planes for Corner Polyhedra, and therefore for integer programming problems, of any size.

2. Application of Corner Polyhedra to generate cutting planes

Now let us turn to generating cutting planes. *The key to this is using the fractional rows, not as cutting planes, but as group characters.* This concept was introduced in [1] Section E. However the subsequent work in [2, 3] has made it simple and straightforward to use this approach without any knowledge of the group structure or of the determinant of the optimal basis for the problem being solved.

2.1. *Group characters, subadditive functions, and cutting planes*

A group character χ is any addition preserving mapping of a group into the real numbers Mod 1. If we take as our group the group generated by addition from the columns $\delta_i = B^{-1}N_i$ of $B^{-1}N$, then such a mapping $\chi_k(\delta_i)$ is given by simply taking the fractional parts of the kth row element in each column and interpreting it as a group element in the interval [0,1) Mod 1. So $\chi_k(\delta_i) = F(\delta_{i,k})$, where $\delta_{i,k}$ is the element in the kth row of the transformed non-basis and F is the fractional part. This mapping χ of vectors into group elements is addition preserving.

Taking any of our many possible row mappings χ_k as our mapping χ, we have from (2)

$$\sum_i F(\delta_{k,i})t_i = F(\delta_{k,0}). \tag{6}$$

If we had a function $\pi(u)$ that was subadditive on the reals Mod 1, we would have, using the subadditivity,

$$\sum_i \pi(F(\delta_{k,i}))t_i \geq \pi(F(\delta_{k,0})). \tag{7}$$

This is a new inequality involving the variables t_i. Also, this is a cutting plane that cuts off the present linear programming vertex whenever $\delta_{k,0}$ is not itself an integer. If $\delta_{k,0}$ is non-integer, then $\chi(\delta_{k,0}) = F(\delta_{k,0}) \neq 0$ and the t_i do not satisfy the inequality (7) at the present linear programming vertex, where they are all 0.

This is a simple and straightforward way to generate cutting planes for any integer program *provided that we have the subadditive functions* $\pi(u)$ *on the reals Mod 1.* However we will see that what we have done so far provides us with a wealth of such subadditive functions. In [3] we show that we have many subadditive functions, and that we can classify them in order of increasing strength as subadditive, minimal, and extreme. The subadditive functions we will be using in our examples here are always minimal and sometimes extreme.

2.2. *Subadditive functions*

Subadditivity on the group elements is a property of any facet π of $P(G, g_0)$ so we always have $\pi(g_1) + \pi(g_2) \geq \pi(g_1 + g_2)$. For cyclic groups, which can be represented as points on the interval [0,1), Mod 1, any such π is subadditive on those points. For a large cyclic group this is close to being subadditive on the reals Mod 1. This suggests that one way to generate subadditive functions is to *interpolate* between the values given to cyclic group elements by any of the facets of any of the Master Polyhedra.

The Straight Line Interpolation Theorem ([3] Theorem 1.9 or [2] Theorem 3.1) asserts that all those connected diagrams that we saw above are subadditive on [0,1). More precisely, if we have a facet π of $P(G, g_0)$ where G is cyclic, we can always form the associated diagram like those shown in Figures 5–12 by connecting the values π_i given to the group elements g_i by straight line segments. If G has n elements, this connected set of lines gives us a function $\pi^*(x)$ on the interval $[0, n)$. We can change $\pi^*(x)$ into a function $\pi(u)$ on there interval $[0, 1)$ by rescaling to the interval [0,1), i.e. $\pi(u) = \pi^*(nu)$. The Straight Line Interpolation Theorem asserts that this $\pi(u)$ is subadditive.

This is one of several ways to generate subadditive functions. There are other ways, one is the "two-slope fill in" [3]. Then there are facets of the Master Polyhedron $P(G, g_0)$ associated with the group G, when G is not a finite group but rather the reals Mod 1, [7]. However, the point is that there is no shortage of subadditive functions and therefore of ways to generate cutting planes for integer programming problems of any size.

2.3. *An example*

Now we give a concrete example of the interpolation procedure. In this example (Table 4) we look at a transformed ith row of a matrix having 15 non-basic variables. The

Table 4. Cutting Plane Example

Fractional coefficients in the row															
0.17	0.21	0.36	0.41	0.44	0.47	0.51	0.59	0.61	0.67	0.71	0.72	0.77	0.81	0.94	0.88
Cutting Plane Coefficients															
0.38	0.12	0.50	0.12	0.50	0.87	1.13	0.13	0.13	0.88	1.13	1.00	0.38	0.13	0.75	1.00
0.45	0.55	0.95	0.95	0.63	0.32	0.03	0.24	0.29	0.45	0.55	0.58	0.71	0.82	0.63	1.00
0.63	0.56	0.85	1.04	0.69	0.35	0.08	0.69	0.75	0.63	0.56	0.54	0.44	0.46	0.69	1.00
0.19	0.24	0.41	0.47	0.50	0.53	0.58	0.67	0.69	0.76	0.81	0.82	0.88	0.92	0.61	1.00
0.80	0.88	0.33	0.36	0.50	0.64	0.78	0.73	0.67	0.33	0.20	0.25	0.48	0.67	0.66	1.00
0.80	0.91	0.44	0.36	0.50	0.64	0.75	0.50	0.44	0.25	0.20	0.25	0.48	0.67	0.66	1.00
0.90	0.86	0.70	0.64	0.61	0.58	0.53	0.45	0.42	0.36	0.32	0.30	0.25	0.32	0.72	1.00

elements shown in the highest row in the table and referred to there as the "Fractional coefficients in the row" are the fractional parts $F(\delta_{i,j})$ of the actual row elements $\delta_{i,j}$.

We have arranged the columns so that these row elements are in order of increasing fractional part. The right hand side element (.88) is in the last column. The first cutting plane generated, which is represented by the topmost of the group of seven rows under Cutting Plane Coefficients in Table 4, is:

$$\begin{aligned}0.38t_1 + 0.12t_2 + 0.50t_3 + 0.12t_4 + 0.50t_5 + 0.87t_6 + 1.13t_7 + 0.13t_8 + 0.13t_9 \\ +0.88t_{10} + 1.13t_{11} + 1.00t_{12} + 0.38t_{13} + 0.13t_{14} + 0.75t_{15} \geq 1\end{aligned}$$

We obtained these numbers by

- Obtaining the first facet of $P(G_{10}, 9)$
- Forming the interpolated function by linear interpolation and then scaling to the interval $[0, 1)$ to obtain a subadditive $\pi(u)$
- Applying that $\pi(u)$ to all the fractional parts $F(c_{i,j})$ in the transformed row just as in equation (7). This produces a cutting plane.
- Dividing all the coefficients through by $\pi(0.88)$ to produce a cutting plane with right hand side $= 1$.

The other rows, each of which represents a cutting plane, were obtained by using the next 6 facets of $P(G_{10}, 9)$ and repeating this procedure.

The reason for the choice of the particular Master Polyhedron $P(G_{10}, 9)$ was to have the peak of the interpolated curve at 9 in the interval $[0, 10)$, which translates into a peak for $\pi(u)$ at 0.9, which is near the right hand side value of 0.88. This tends to make almost all the coefficients less than the right hand side, which makes for a stronger inequality. We could equally well have chosen $P(G_{11}, 10)$, which would give a peak at 0.91, or $P(G_{20}, 18)$, or $P(G_8, 7)$, or $P(G_{16}, 14)$. Or we could have chosen them all and generated large numbers of cutting planes.

The seven cutting planes we actually computed form a well balanced set. Some cut deeply on one set of variables, others cut deeply on other variables. This is what we would expect as these inequalities come from the major facets of a Corner Polyhedron.

2.4. *Remarks on Non-Master Polyhedra*

The theory of [1] shows that the facets of a particular Corner Polyhedron are among the facets of the Master Polyhedron. That is, if we set $t = 0$ for the group elements that are not present, the resulting inequalities include all the facets of the particular Corner Polyhedron $P(G, N, g_0)$. It seems plausible that in problems with many variables, there will be a good coverage of group elements present on the unit interval (the rescaled interval $[0, n)$). Major inequalities from the Master Polyhedra are likely to be relevant.

For problems with a spotty coverage of the real line, we should be able to use the great variety of possible inequalities and their systematic structure to shape strong inequalities, or sets of strong inequalities, to deal with the actual locations of the fractional parts.

3. General remarks on the use of Corner Polyhedra and moving beyond cutting planes

Clearly there is much to be learned about Corner Polyhedra, and that knowledge translates directly into cutting planes. The use of fractional rows of the updated linear programming tablueau as group characters allows us to import sets of valid inequalities from Corner Polyhedra or from our knowledge of extreme functions on the interval $[0, 1]$ ([7]) and translate them into cutting planes.

Practice seems to show the effectiveness of adding multiple cutting planes to a formulation before resolving the linear programming relaxation. From our point of view, this makes sense, since adding a strong, well-balanced set of facets from appropriate Corner Polyhedra begins to approximates the true Corner Polyhedron, rather than making a single cut and moving to a new linear programming vertex. One approach used in practice today is to add many Gomory cuts from different rows of the updated tableau in the same iteration. Adding many such inequalities from different rows and with different right hand side elements (although this is not obvious) is, in the case of a pure integer program, almost exactly the same as using the automorphic images of many different Gomory cuts on a single row of the tableau. This then ends up being a simple way to generate and apply one important subset of the Master Polyhedra facets.

If we were able to come close to solving the Corner Polyhedron problem – say by having an adequate supply of cutting planes or perhaps in other ways, such as finding solutions to the group problem, we could come close to a different kind of algorithm – one based on solving a sequence of Corner Polyhedron problems. In such an algorithm, we would solve a corner problem, then apply dual simplex to the resulting integer answer, and then repeat. With the practice of putting on multiple cutting planes, we are moving in that direction.

A. Faces tilted away from the origin

We have examined the question of whether facets tilted away from the origin receive fewer hits in the shooting experiment in two ways. (1) by looking at the normals of

Table 5. Cosine test tables

Number of hits	Cosine of angle with (1,1,...,1)	Cosine of angle with incidence direction	Number of hits	Cosine of angle with (1,1,...,1)	Cosine of angle with incidence direction
20	0.802	0.740	2	0.904	0.617
9	0.845	0.662	1	0.934	0.821
8	0.888	0.764	1	0.934	0.807
8	0.861	0.597	1	0.931	0.797
7	0.900	0.617	1	0.926	0.770
6	0.891	0.882	1	0.925	0.759
4	0.915	0.793	1	0.923	0.748
4	0.909	0.624	1	0.922	0.743
3	0.905	0.770	1	0.922	0.710
3	0.892	0.706	1	0.920	0.699
3	0.881	0.680	1	0.912	0.690
2	0.926	0.826	1	0.911	0.665
2	0.925	0.814	1	0.911	0.654
2	0.916	0.799	1	0.911	0.651
2	0.908	0.678	1	0.901	0.635

the facets that are hit to see how far off the direction (1, 1, . . . , 1) they are, and (2) by looking at the angle of the random direction that hit the facet made with the normal to that facet.

The result of examining one Master Polyhedron, $P(G_{15}, 14)$ is shown in Figure A. The lists in the figure are the facets arranged in order of decreasing hits. The first number in each row is the number of hits out of 99. In the first table the second column is the cosine of the angle made with (1, 1, . . . , 1). In the second table the second number is the cosine of the angle made with the incident random direction. There seems to be at most a very slight tendency for the low hit facets to be parallel to the axes or hit at a sharper angle, but certainly nothing that would explain the significant disparity in hits. Similar examinations have been made for several other Master Polyhedra and the results are always the same.

References

1. Gomory, R.E.: Some polyhedra related to combinatorial problems. Linear Algebra App. **2**, 451–558 (1969)
2. Gomory, R.E., Johnson, E.L.: Some continuous functions related to corner polyhedra, I. Math. Programming **3**, 23–85 (1972)
3. Gomory, R.E., Johnson E.L.: Some continuous functions related to corner polyhedra, II. Math. Programming **3**, 359–389 (1972)
4. Gilmore, P.C., Gomory, R.E.: The theory and computation of knapsack functions. Operations Res. **14**(6), 1045–1074 (1966)
5. Lenstra, J.K., Rinnooy Kan, A.H.G., Schrijver, A.: On the origin of the hungarian method History of Mathematical Programming: A Collection of Personal Reminiscences, pp 77–81. CWI and North Holland, Amsterdam, The Netherlands, 1991. *Harold W. Kuhn*
6. Aráoz, J., Evans, L., Gomory, R., Johnson, E.: Cyclic group and knapsack facets. Math. Programming the same issue, 2003
7. Gomory, R., Johnson, E.: T-space and cutting planes. Math. Programming the same issue, 2003