SOME PROPERTIES OF THE RANK AND INVARIANT FACTORS OF MATRICES*

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1. Introduction. Some years ago, G. Pall observed that the invariant factors of the incidence matrices of a certain pair of non-isomorphic projective planes of order 9 were different. With the aim of investigating such phenomena experimentally, we have constructed a code to calculate the invariant factors of rational integral matrices (actually, we compute the Smith's normal form of these matrices, as described, e.g., in MacDuffee [2; p. 41]), and this note is in the nature of a report on some preliminary experiments in the use of this code. In particular, we have computed the invariant factors of all (0, 1) matrices of order ≤ 8, with constant row and column sums, and these data are presented in the Appendix.

An examination of these data suggested three conjectures, all of which turned out to be true, and one of which suggested some interesting questions concerning the imbedding of a non-singular matrix in a doubly stochastic matrix of the same rank. The proofs of the conjectures (Remark 1, Remark 2 and Theorem 2) and the discussion of the questions of imbedding (Theorem 1, Theorem 3 and Theorem 4) form the main part of the note. We hope that others may discern additional facts from the data tabulated in the Appendix.

2. Some Simple Remarks. Let $\mathcal{O}\!\mathcal{l}$ be the class consisting of all $m \times n$ (0, 1) matrices with prescribed row sums and column sums.

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Remark 1. The ranks of matrices in $\mathfrak{A}$ assume all integers between the minimum rank and the maximum rank of matrices in $\mathfrak{A}$.

Proof: Let A be a matrix in $\mathfrak{A}$. Consider the 2×2 submatrices of A of the types

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

An <u>interchange</u> is a transformation of the elements of A that changes a minor of type A_1 into type A_2 or vice versa and leaves all other elements of A unaltered. The interchange theorem of H. J. Ryser [3] states that if A and B belong to $\mathfrak{A}$, then A is transformable into B by a finite sequence of interchanges. We note that if B is a matrix obtained from A by an interchange, then $B = A + C$ where C is a matrix whose entries are all zero except for a minor of the form $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, and therefore rank $B \leq$ rank $A + 1$. Similarly, rank $A \leq$ rank $B + 1$. Hence, the rank is altered by at most 1 by an interchange, and the statement to be proven follows at once by the interchange theorem of Ryser.

From now on, let J be a matrix of the appropriate size whose entries are all 1, and let $\tilde{A} = J - A$.

Remark 2. Let A be any $n \times n$ matrix with rational integral entries. Then the number of units among the invariant factors of A and the number of units among the invariant factors of $\tilde{A}$ differ by at most 1.

Proof: Let A_1 be the matrix obtainable from A by subtracting the first column from every other column, and let $\tilde{A}_1$ be the matrix obtainable from $\tilde{A}$ correspondingly. Except possibly for the first columns, A_1 and $\tilde{A}_1$ are the same except for sign, and therefore if $e_1, e_2, \ldots, e_n$, with $e_1 | e_2 | \ldots | e_n$, are the invariant factors of A_1 and $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n$, with $\tilde{e}_1 | \tilde{e}_2 | \ldots | \tilde{e}_n$, are the invariant factors of $\tilde{A}_1$, then

$$\left(\prod_{m=1}^{k} e_m\right) \Bigg| \left(\prod_{m=1}^{k+1} \tilde{e}_m\right) \text{ and } \left(\prod_{m=1}^{k} \tilde{e}_m\right) \Bigg| \left(\prod_{m=1}^{k+1} e_m\right), \quad k = 1, 2, \ldots, n-1.$$

Suppose e_i is the first non-unit invariant factor of A_1 and $\tilde{e}_j$ is the first non-unit invariant factor of $\tilde{A}_1$. We may as well assume that $i \leq j$ and $j > 2$. Taking $k = j - 2$, we have

$$e_{j-2} \Bigg| \left(\prod_{m=1}^{j-2} e_m\right) \Bigg| \left(\prod_{m=1}^{j-1} \tilde{e}_m\right) = \pm 1$$

and therefore $j - 1 \leq i \leq j$, which completes the proof.

3. Imbedding Questions. Let $\mathcal{A}(n, k, p)$ denote the class of all $n \times n$ matrices with real entries whose row and column sums are all $k \neq 0$, and whose rank is $p \leq n$. We shall write $B < \mathcal{A}(n, k, p)$ for a non-singular matrix B of order p if there exists a matrix $A \in \mathcal{A}(n, k, p)$ which contains B as a submatrix. We derive as Theorem 1 a necessary condition for $B < \mathcal{A}(n, k, p)$. As a corollary, we prove a relation between the invariant factors of a rational integral square matrix A and the invariant factors of $J - A$. In Theorem 4, we show that this necessary condition is also sufficient to imbed a rational integral matrix B in a rational integral matrix $A \in \mathcal{A}(n, k, p)$. We also show in Theorem 3 a set of necessary and sufficient conditions for imbedding a non-negative matrix B in a non-negative matrix $A \in \mathcal{A}(n, 1, p)$, i. e., a doubly stochastic matrix of rank p.

THEOREM 1. If $B < \mathcal{A}(n, k, p)$, then the sum of the elements of B^{-1} is $\frac{n}{k}$.

Proof: Assume

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix},$$

where $A \in \mathcal{O}(n,k,p)$. Since A is of rank p,

$$(3.1) \qquad E = D B^{-1} C.$$

Let u_t be the column vector with t coordinates, each of which is unity. Since the row sums of A are k, we have

$$(3.2) \qquad B u_p + C u_{n-p} = k u_p ,$$

$$(3.3) \qquad D u_p + E u_{n-p} = k u_{n-p} .$$

Inserting (3.1) and (3.2) in (3.3), we obtain

$$(3.4) \qquad D B^{-1} u_p = u_{n-p} .$$

Since the column sums of A are k, we have

$$(3.5) \qquad u'_p B + u'_{n-p} D = k u'_p .$$

Multiplying both sides of (3.5) on the right by $B^{-1} u_p$, and substituting in (3.4), we obtain

$$p + n - p = k u'_p B^{-1} u_p ,$$

which was to be proved.

THEOREM 2. Let A be a matrix of order n, with row and column sums k, $0 \neq k \neq n$, whose entries are rational integers, and let $\tilde{A} = J - A$. Let $\tilde{e}_1, \ldots, \tilde{e}_n$ be the invariant factors of $\tilde{A}$; $e_1, \ldots, e_n$ the invariant factors of A. Then

$$(3.6) \qquad \frac{\prod_{e_i \neq 0} e_i}{\prod_{\tilde{e}_i \neq 0} \tilde{e}_i} = \frac{k}{n-k}$$

up to a unit ± 1.

Proof: Let A be of rank p, B a non-singular matrix of order p contained in A, and $\tilde{B} = J - B$. We first show that the rank of $\tilde{B}$ is equal to the rank of $\tilde{A}$ and that $|\tilde{B}| \neq 0$. Write

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}, \quad \tilde{A} = J - A = \begin{pmatrix} \tilde{B} & \tilde{C} \\ \tilde{D} & \tilde{E} \end{pmatrix}.$$

Since the column sums of A are all k, we have from (3.1)

$$u'_p C + u'_{n-p} D B^{-1} C = k u'_{n-p} . \tag{3.7}$$

Substituting from (3.5), we have

$$u'_p B^{-1} C = u'_{n-p} . \tag{3.8}$$

Let X be a matrix with p rows and $n-p$ columns such that

$$BX = C . \tag{3.9}$$

From (3.8), we have

$$u'_p X = u'_{n-p} . \tag{3.10}$$

It is then clear from (3.9) and (3.10) that

$$\tilde{B} X = \tilde{C} . \tag{3.11}$$

Further, since $DX = E$, from (3.1), it follows from (3.10) that

$$\tilde{D} X = \tilde{E} . \tag{3.12}$$

In other words, the last $n-p$ columns of $\tilde{A}$ are linear combinations of the first p columns. Similarly, we can see that the last $n-p$ rows of $\tilde{A}$ are linear combinations of the first p rows. Consequently, rank $\tilde{A}$ = rank $\tilde{B} \leq p$ = rank B = rank A. But, symmetrically, rank $A \leq$ rank $\tilde{A}$, which implies rank A = rank $\tilde{A}$ and therefore rank $\tilde{B} = p$, from which $|\tilde{B}| \neq 0$ follows. Now, we have from Theorem 1 that

$$u'_p B^{-1} u_p = \frac{n}{k}, \quad u'_p \tilde{B}^{-1} u_p = \frac{n}{n-k} . \tag{3.13}$$

We shall use (3.13) to prove that

(3.14) $$\frac{|B|}{|\tilde{B}|} = (-1)^{p-1}\frac{k}{n-k}.$$

Clearly, (3.14) implies (3.6), for the numerator of the left-hand side of (3.6) is the g.c.d. of the determinants of order p contained in A, and the denominator is the g.c.d. of the corresponding determinants in $\tilde{A}$. To prove (3.14), observe first that

(3.15) $$u_p' B^{-1} u_p = u_p' (B^{-1} u_p) = \frac{1}{|B|} \sum_j \Delta_j ,$$

where Δ_j is the determinant of the matrix whose kth column, $k \neq j$, is B_k, the kth column of B, and whose jth column is u_p. Now,

(3.16) $$\sum_j \Delta_j = |u_p, B_2 - B_1, \ldots, B_p - B_1| ,$$

which can be verified from the expansion of the right-hand side. Further, if we apply the same observations to $\tilde{B}$, we have

(3.17) $$u_p' \tilde{B}^{-1} u_p = \frac{1}{|\tilde{B}|} \sum_j \tilde{\Delta}_j ,$$

where $\tilde{\Delta}_j$ is defined in the obvious way, and

(3.18) $$\sum_j \tilde{\Delta}_j = |u_p, \tilde{B}_2 - \tilde{B}_1, \ldots, \tilde{B}_p - \tilde{B}_1| .$$

But $\tilde{B}_k - \tilde{B}_1 = -(B_k - B_1)$, and, therefore, from (3.18) and (3.16), we have

(3.19) $$\sum_j \tilde{\Delta}_j = (-1)^{p-1} \sum_j \Delta_j .$$

Finally, (3.14) follows at once from (3.19), (3.17), (3.15) and (3.13).

THEOREM 3. Let B be a non-singular matrix of order p whose entries are non-negative real numbers. In order that B be a submatrix of a doubly stochastic matrix A of order n and rank p, where $p < n \leq 2p-1$, it is necessary and sufficient that

(3.20) $$u_p' B^{-1} u_p = n ,$$

(3.21) $$\sum_j b_{ij} \leq 1, \quad i = 1, \ldots, p; \quad \sum_i b_{ij} \leq 1, \quad j = 1, \ldots, p ,$$

(3.22) $$\sum_{i,j} b_{ij} \geq 2p - n .$$

Proof. The necessity of (3.20) is contained in Theorem 1. The necessity and sufficiency of (3.21) and (3.22) in order to effect the imbedding in a doubly stochastic matrix without considering the rank was pointed out by Dulmage and Mendelsohn [1]. To prove the sufficiency of (3.20) - (3.22) with the rank taken into consideration, write

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix} ,$$

where C, D and E are to be determined. Let each column of C be $\frac{1}{n-p}(I-B)u_p$, each row of D be $\frac{1}{n-p} u_p' (I-B)$, and each entry of E be $\frac{1}{(n-p)^2}(\sum_{i,j} b_{ij} + n - 2p)$. Then the non-negativity of the entries of A follows from (3.21) and (3.22), and that A is doubly stochastic is easy to verify. To show that the rank of A is equal to the rank of B, we must see that

$$\frac{1}{(n-p)^2}(u_p' B u_p + n - 2p) = \frac{1}{n-p} u_p' (I-B) B^{-1} \frac{1}{n-p} (I-B) u_p ,$$

which follows from (3.20).

THEOREM 4. If B is a non-singular matrix of order p with rational integral entries, and the sum of the coefficients of B^{-1} is $\frac{n}{k}$, $n > p$, then B is contained in a matrix of order n and rank p, with rational integral entries, whose row and column sums are all k.

Proof: We wish to find a matrix

$$A = \begin{pmatrix} B & & C \\ & F & \end{pmatrix}$$

with the desired properties. Let all columns of C other than the last be the same as columns of B, and choose the last column in such a way that the row sums, for each of the first p rows, shall be k. Next, let all rows of F other than the last be the same as rows of (B, C). Choose the last row of F in such a way that all column sums of A are k. All that needs to be checked is that the rank of A is p, which can be done as in the previous theorem.

REFERENCES

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APPENDIX

How to read the table:

$$(n, k) \quad \rightarrow \quad (n, n-k)$$

$$g_1, g_2, \ldots, g_n \qquad\qquad h_1, h_2, \ldots, h_n$$

means that if A, a $n \times n$ matrix of zeros and ones (whose row sums and column sums are all k) has invariant factors $g_1, g_2, \ldots, g_n$, then $J-A$ has invariant factors $h_1, h_2, \ldots, h_n$.

$(n, k) \rightarrow (n, n-k)$

$$g_1, g_2, \ldots, g_n \qquad \begin{cases} h_1, h_2, \ldots, h_n \\ h'_1, h'_2, \ldots, h'_n \end{cases}$$

means that if A is the same matrix as described above, then J-A has invariant factors either $h_1, h_2, \ldots, h_n$ or $h'_1, h'_2, \ldots, h'_n$.

(4,1)				⇄	(4,3)			
1	1	1	1		1	1	1	3
(4,2)				⇄	(4,2)			
1	1	0	0		1	1	0	0
1	1	1	0		1	1	1	0

(5,1)					⇄	(5,4)				
1	1	1	1	1		1	1	1	1	4
(5,2)					⇄	(5,3)				
1	1	1	2	0		1	1	1	3	0
1	1	1	1	2		1	1	1	1	3

(6,1) ⇄ (6,5)

1	1	1	1	1	1	⁞	1	1	1	1	1	5

(6,2) ⇄ (6,4)

1	1	1	0	0	0	⁞	1	1	2	0	0	0
1	1	1	1	0	0	⁞	1	1	1	2	0	0
1	1	1	1	1	0	⁞	1	1	1	1	2	0
1	1	1	1	2	2	⁞	1	1	1	1	1	8

(6,3) ⇄ (6,3)

1	1	0	0	0	0	⁞	1	1	0	0	0	0
1	1	1	0	0	0	⁞	1	1	1	0	0	0
1	1	1	1	0	0	⁞	1	1	1	1	0	0
1	1	1	1	1	0	⁞	1	1	1	1	1	0
1	1	1	1	1	9	⁞	1	1	1	1	1	9

(7,1) ⇄ (7,6)

1	1	1	1	1	1	1	⁞	1	1	1	1	1	1	6

(7,2) ⇄ (7,5)

1	1	1	1	2	0	0	⁞	1	1	1	1	5	0	0
1	1	1	1	1	2	0	⁞	1	1	1	1	1	5	0
1	1	1	1	1	1	2	⁞	1	1	1	1	1	1	5

(7,3) ⇄ (7,4)

1	1	1	1	3	0	0	⁞	1	1	1	1	4	0	0
1	1	1	1	1	3	0	⁞	1	1	1	1	1	4	0
1	1	1	1	1	1	3	⁞	1	1	1	1	1	1	4
1	1	1	1	2	2	6	⁞	1	1	1	2	2	2	4

(8,1) ⇄ (8,7)

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	7

(8,2) ⇄ (8,6)

1	1	1	1	0	0	0	0	1	1	1	3	0	0	0	0
1	1	1	1	1	0	0	0	1	1	1	1	3	0	0	0
1	1	1	1	1	1	0	0	1	1	1	1	1	3	0	0
1	1	1	1	1	1	1	0	1	1	1	1	1	1	3	0
1	1	1	1	1	2	2	0	1	1	1	1	1	1	12	0
1	1	1	1	1	1	2	2	1	1	1	1	1	1	1	12

(8,3) ⇄ (8,5)

1	1	1	1	3	0	0	0	1	1	1	1	5	0	0	0
1	1	1	1	1	3	0	0	1	1	1	1	1	5	0	0
1	1	1	1	1	1	3	0	1	1	1	1	1	1	5	0
1	1	1	1	1	1	6	0	1	1	1	1	1	1	10	0
1	1	1	1	1	1	1	3	1	1	1	1	1	1	1	5
1	1	1	1	1	1	1	6	1	1	1	1	1	1	1	10
1	1	1	1	1	1	1	15	1	1	1	1	1	1	5	5
1	1	1	1	1	1	3	3	1	1	1	1	1	1	1	15
1	1	1	1	1	3	3	3	1	1	1	1	1	1	3	15

(8, 4)								→ (8, 4)							
1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0
1	1	1	0	0	0	0	0	1	1	1	0	0	0	0	0
1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
1	1	1	2	0	0	0	0	1	1	1	2	0	0	0	0
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1	1	1	1	2	0	0	0	1	1	1	1	2	0	0	0
1	1	1	1	1	1	0	0	1	1	1	1	1	1	0	0
1	1	1	1	1	2	0	0	1	1	1	1	1	2	0	0
1	1	1	1	1	4	0	0	1	1	1	1	1	4	0	0
								1	1	1	1	2	2	0	0
1	1	1	1	2	2	0	0	1	1	1	1	1	4	0	0
1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	0
1	1	1	1	1	1	2	0	1	1	1	1	1	1	2	0
1	1	1	1	1	1	4	0	1	1	1	1	1	1	4	0
								1	1	1	1	1	2	2	0
1	1	1	1	1	1	8	0	1	1	1	1	1	1	8	0
								1	1	1	1	1	2	4	0
1	1	1	1	1	1	16	0	1	1	1	1	1	4	4	0
1	1	1	1	1	2	2	0	1	1	1	1	1	1	4	0
1	1	1	1	1	2	4	0	1	1	1	1	1	1	8	0
1	1	1	1	1	4	4	0	1	1	1	1	1	1	16	0
1	1	1	1	1	1	1	16	1	1	1	1	1	1	2	8
								1	1	1	1	1	1	4	4
1	1	1	1	1	1	1	32	1	1	1	1	1	1	1	32
1	1	1	1	1	1	2	8	1	1	1	1	1	1	1	16
1	1	1	1	1	1	4	4	1	1	1	1	1	1	1	16

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