Reprinted from OPERATIONS RESEARCH
Vol. 12, No. 5, September-October, 1964
Printed in U.S.A.

September–October 1964

SEQUENCING A ONE STATE-VARIABLE MACHINE: A SOLVABLE CASE OF THE TRAVELING SALESMAN PROBLEM

P. C. Gilmore and R. E. Gomory†

IBM Watson Research Center, Yorktown Heights, New York

(Received January 28, 1964)

We consider a machine with a single real variable x that describes its state. Jobs $J_1, \cdots, J_N$ are to be sequenced on the machine. Each job requires a starting state A_i and leaves a final state B_i. This means that J_i can be started only when $x=A_i$ and, at the completion of the job, $x=B_i$. There is a cost, which may represent time or money, etc., for changing the machine state x so that the next job may start. The problem is to find the minimal cost sequence for the N jobs. This problem is a special case of the traveling salesman problem. We give a solution requiring only $0(N^2)$ simple steps. A solution is also provided for the bottleneck form of this traveling salesman problem under special cost assumptions. This solution permits a characterization of those directed graphs of a special class which possess Hamiltonian circuits.

WE WILL consider the sequencing problem for N jobs on a machine having a state described by a single real variable x.‡ Jobs $J_1, \cdots, J_N$, are to be done on the machine in some order. Each job has two associated numbers A_i and B_i. To start the ith job, the machine must be in state A_i, i.e., $x=A_i$, and at the completion of the ith job the machine state is automatically B_i. If J_j is to follow J_i, the state of the machine must then be changed to A_j. The cost of this change is c_{ij}, the cost of having job j follow job i. This is taken to be

$$\begin{cases} c_{ij}=\int_{B_i}^{A_j} f(x)\,dx & \text{if } A_j \geqq B_i, \\ c_{ij}=\int_{A_j}^{B_i} g(x)\,dx & \text{if } B_i > A_j. \end{cases} \tag{1}$$

† This research was supported in part by the Office of Naval Research under Contract No. Nonr 3775(00), NR 047040.

‡ An announcement of a special case of the results of this paper appears in reference [2].

Where $f(x)$ and $g(x)$ are any integrable functions satisfying

$$f(x)+g(x)\geqq 0, \tag{2}$$

$f(x)$ can be interpreted as the cost density for increasing the state variable, and $g(x)$ the cost density for decreasing it. Restriction (2) implies that there is no gain to be obtained by cycling the state variable, i.e., by increasing it and then decreasing it to its original state.

In this paper, we give a method for finding the ordering of the J_i that minimizes the total changeover cost. We also give a method for finding the ordering of the J_i that minimizes the maximum changeover cost when $g(x)=0$ and $f(x)\geqq 0$.

To make the problem more concrete let us consider an example. Consider a furnace and let the state variable be the temperature. Various jobs are to be put through a temperature cycle inside the furnace. The ith job, J_i, will be started at temperature A_i and taken out of the furnace at temperature B_i. The temperature is then changed for the next job. There is a cost $f(x)$ for heating the furnace one degree and a cost $g(x)$ for cooling it one degree when the temperature is x. We are looking for the least cost sequence of the jobs.

Our problem is of course closely connected to the well-known and difficult Traveling Salesman problem.[1, 5, 8] To see this, let the J_i play the role of nodes or cities, and let the c_{ij} of (1) be the cost of going from node i to node j. We are looking for the cheapest path that passes once through each node. The traveling salesman looks for the cheapest path that passes once through each node and ends up at the starting point. He looks for the cheapest tour.

Our sequencing problem becomes the tour problem if the machine is assumed to be in a state B_0 at the start, before the jobs are run, and is required to be left in a state A_0 at the end after all N jobs are done. Specifically, if we add a new job J_0, there is always a one-to-one correspondence between the tours $J_0J_{i_1}J_{i_2}\cdots J_{i_N}J_0$ of the enlarged problem and the sequences $J_{i_1}J_{i_2}\cdots J_{i_N}$ of the original problem. Thus we can minimize over tours, and then, dropping J_0, have the least cost sequence.

As it is a considerable technical advantage to deal with tours rather than sequences, we will deal with tours from now on. So we will, implicitly, be dealing with a sequencing problem with a prescribed initial and final state. However, in the final section we will show how, in some cases, a tour solution also solves the sequencing problem in which no starting or ending states are required.

From this point on we will discuss only the minimal cost tour problem. We will state the problem in terms of a permutation ψ for which the total changeover cost $c(\psi)$,

$$c(\psi)=\sum_{i=1}^{i=N} c_{i\psi(i)}, \tag{3}$$

is minimal subject to the condition that ψ gives a tour, i.e., subject to

$$\psi(s)\neq s \tag{4}$$

for all proper subsets s of the numbers $1, 2, \cdots, n$.

Our method for solving this problem is roughly the following. First, find the permutation φ that minimizes (3) disregarding (4). Then by carrying out a series of interchanges, we convert the permutation φ into a tour ψ. The interchanges to be executed will be chosen by finding a minimal spanning tree and must be carried out in a special order if the resulting tour ψ is to be minimal.

In the next section we describe the type of interchange used and its effect on the cost function (3). We also find the minimal cost permutation. In the third section we describe the connection between the tour and spanning tree problems.

In the fourth section, which is merely preparation for the fifth, we describe a special property that the spanning trees possess. In the fifth section we determine a set of interchanges and the proper order to execute them to obtain a low cost tour ψ^*.

In the following two sections we prove that the tour ψ^* obtained in the fifth section is in fact a minimal tour.

In the eighth section we treat the 'bottleneck' Traveling Salesman problem. This is the problem of minimizing the largest changeover cost occurring in the tour.

The last section, contains a complete statement of the algorithm followed by a numerical example. There is an application to a cutting problem, various remarks on and extensions to the earlier sections, and a brief discussion of the relation between the problem described here and the well-known work of S. M. JOHNSON.[6]

INTERCHANGES AND THEIR COSTS

IN DESCRIBING permutations, interchanges, and their costs, it is useful to introduce a diagram displaying the B_i and A_i. We will assume that the B_i are so numbered that $j>i$ implies $B_j \geq B_i$, and that they are arranged on a real line in the positions corresponding to their values. Position B_i on this line then gives the value of the state variable at the end of job i. It would be natural to display the A_i on this same line but, in order to show the permutation ψ better, we will devote a separate line to the A_i values (Fig. 1).

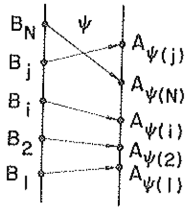

Figure 1

In Fig. 1 ψ is displayed as a series of arrows linking the final state B_i of J_i to the starting state $A_{\psi(i)}$ of the succeeding job $J_{\psi(i)}$. If in Fig. 1 $A_{\psi(i)}>B_i$, then the ith arrow goes up and the first formula of (1) is used in computing $c_{i\psi(i)}$. If the ith arrow goes down the second one is used.

We will now define an interchange and compute its effect on the cost $c(\psi)$.

By an interchange α_{ij} we mean the permutation given by

$$\begin{cases}\alpha_{ij}(i)=j,\\ \alpha_{ij}(j)=i, \\ \alpha_{ij}(k)=k. & (k\neq i,j)\end{cases} \tag{5}$$

Applying an interchange to ψ yields a new permutation $\bar{\psi}$ given by

$$\bar{\psi}=\psi\alpha_{ij}, \tag{6}$$

i.e., first apply α_{ij}, then ψ to get $\bar{\psi}$. Clearly

$$\begin{cases}\bar{\psi}(i)=\psi(j),\\ \bar{\psi}(j)=\psi(i), \\ \bar{\psi}(k)=\psi(k). & (k\neq i,j)\end{cases} \tag{7}$$

So the effect of α_{ij} on ψ is to interchange the successors of i and j, (Fig. 2).

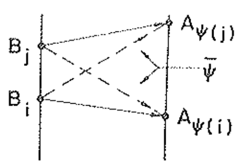

Figure 2

We define the cost $c_\psi(\alpha_{ij})$ of applying α_{ij} to ψ by

$$c_\psi(\alpha_{ij})=c(\psi\alpha_{ij})-c(\psi).$$

We will give a formula for $c_\psi(\alpha_{ij})$. In this formula we will refer to intervals $[a,b]$, $a\leqq b$, on the real line. For such intervals, we will use the notation

$$\|[a,b]\|_f=\int_a^b f(x)\,dx,$$

$$\|[a,b]\|_g=\int_a^b g(x)\,dx,$$

$$\|\|[a,b]\|\|=\int_a^b \{f(x)+g(x)\}\,dx.$$

Let us assume that

$$\begin{cases} B_j \geqq B_i, \text{ and} \\ A_{\psi(j)} \geqq A_{\psi(i)}. \end{cases} \tag{8}$$

Then, using (1) we have

$$c_{i\psi(j)}=\|[B_i,+\infty]\cap[-\infty,A_{\psi(j)}]\|_f+\|[-\infty,B_i]\cap[A_{\psi(j)},+\infty]\|_g,$$
$$c_{i\psi(i)}=\|[B_i,+\infty]\cap[-\infty,A_{\psi(i)}]\|_f+\|[-\infty,B_i]\cap[A_{\psi(i)},+\infty]\|_g.$$

Subtracting gives

$$c_{i\psi(j)}-c_{i\psi(i)}=\|[B_i,+\infty]\cap[A_{\psi(i)},A_{\psi(j)}]\|_f-\|[-\infty,B_i]\cap[A_{\psi(i)},A_{\psi(j)}]\|_g. \tag{9}$$

Similarly,

$$c_{j\psi(i)}-c_{j\psi(j)}=-\|[B_j,+\infty]\cap[A_{\psi(i)},A_{\psi(j)}]\|_f+\|[-\infty,B_j]\cap[A_{\psi(i)},A_{\psi(j)}]\|_g. \tag{10}$$

Adding (9) and (10) we get

$$c_\psi(\alpha_{ij})=c_{i\psi(j)}+c_{j\psi(i)}-c_{i\psi(i)}-c_{j\psi(j)}=\|\|[B_i,B_j]\cap[A_{\psi(i)},A_{\psi(j)}]\|\|. \tag{11}$$

Since $f(x)+g(x)\geqq 0$, we have $c_\psi(\alpha_{ij})\geqq 0$. (11) was of course obtained under assumption (8). If (8) does not hold and the order of $A_{\psi(j)}$ and $A_{\psi(i)}$ is the reverse of the order of B_j and B_i, then the expression on the right in (11) is preceded by a minus sign.

This gives Theorem 1.

THEOREM 1. *Let φ be a permutation that ranks the A, that is $j>i$ implies $A_{\varphi(j)}\geqq A_{\varphi(i)}$, then*

$$c(\varphi)=\min_\psi c(\psi),$$

with the minimum being taken over all permutations ψ.

Proof. This is clear from Fig. 3, for if a permutation ψ contains a reversal of order between a B_j and B_i and $A_{\psi(j)}$ and $A_{\psi(i)}$ (the crossed arrows of Fig. 3) then the cost of applying α_{ij} is negative or zero. Thus by successive uncrossings any ψ is reduced to the crossing free permutation φ, which has a cost lower or equal to $c(\psi)$.

TOURS AND TREES

LET US consider the effect of an interchange α_{ij} on the cycles in a permutation ψ. The situation is illustrated in Fig. 4 and it is not hard to prove

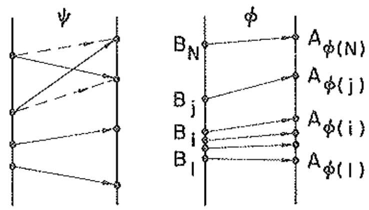

Figure 3

formally what is evident geometrically. That is

LEMMA 1. *If ψ is a permutation consisting of cycles $C_1 \cdots C_p$, and α_{ij} is an interchange with $i \epsilon C_r$ and $j \epsilon C_s, r \neq s$, then $\psi\alpha_{ij}$ contains the same cycles except that C_r and C_s have been replaced by a single cycle containing all their nodes.*

This is geometrically evident from Fig. 4(a). We note that if i and j belong to the same cycle, Fig. 4(b), the effect is reversed and the cycle is split.

We next define a graph G_ψ for any permutation ψ: G_ψ has N nodes and an *undirected* arc linking the ith and $\psi(i)$th nodes, for $i=1, \cdots, N$. Clearly there is a correspondence between the cycles of ψ and the connected components of G_ψ. For each cycle of ψ on a given set of nodes, there is a connected component of G_ψ consisting of the same set of nodes.

This correspondence between the cycles of the permutation and the connected components of the graph can be maintained under certain simple changes. Suppose we add to G_ψ an arc R_{ij} connecting two nodes i and j in different components. The effect is to leave all other components alone and unite the two components involved. Since i and j were in different cycles of ψ, Lemma 1 tells us that the components of $G_\psi \cup R_{ij}$ now correspond to the cycles of $\psi\alpha_{ij}$.

Starting with a graph G_ψ we can add a set of arcs to it until it becomes a connected graph. If G_ψ has p components, the minimal number of arcs required to connect it is always $p-1$. We will call a minimal set of additional arcs that connect a graph G_ψ a 'spanning tree.' Our nomenclature

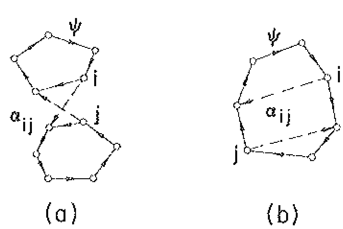

Figure 4

agrees with the usual one of reference 7 if the original components of G_ψ are regarded as points. We can now state Theorem 2.

THEOREM 2. *Let* $\alpha_{i_1 j_1}, \alpha_{i_2 j_2}, \cdots, \alpha_{i_{p-1}\, j_{p-1}}$ *be the interchanges corresponding to the arcs of a spanning tree of* G_ψ. *The arcs may be taken in any order. Then the permutation* ψ' *given by*

$$\psi' = \psi \alpha_{i_1 j_1} \alpha_{i_2 j_2} \cdots \alpha_{i_{p-1}\, j_{p-1}}$$

is a tour.

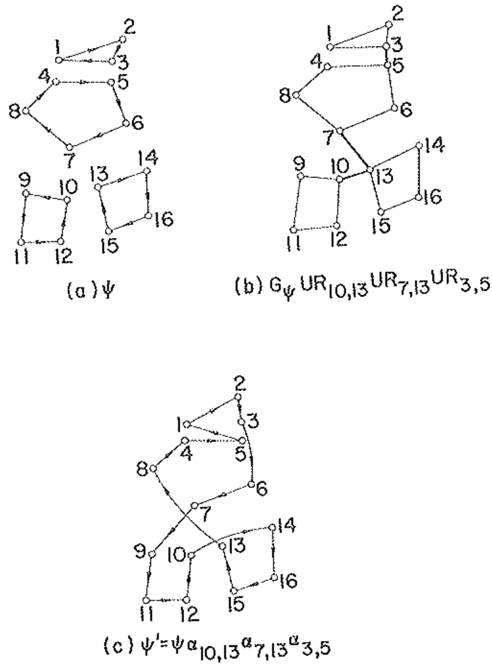

Figure 5

Figure 5 gives an example.

Proof: We need only show that the correspondence between cycles and components is maintained, for since $G_\psi \cup R_{i_1 j_1} \cup R_{i_2 j_2} \cup \cdots \cup R_{i_{p-1}\, j_{p-1}}$ is connected by the definition of spanning tree, this correspondence would allow only one cycle for ψ'. We know that the correspondence is maintained if each added arc connects two components. However since we are using the arcs of a spanning tree this two-component property is automatic. For if one of these arcs, when added, connected nodes of the same component, then the remaining arcs, without that one would already make G_ψ connected, which contradicts the assumed minimality of the connecting set.

TREE COSTS AND A SPECIAL PROPERTY OF THE MINIMAL TREE

HAVING established a connection between trees and tours we next introduce costs on the arcs of the trees.

We start with the graph G_φ formed using the minimal permutation φ of the second section. To any arc R_{ij}, corresponding to the interchange α_{ij}, we assign a cost $c_\varphi(\alpha_{ij})$, the cost of applying the corresponding interchange to φ.

By the cost $c_\varphi(\tau)$ of a tree τ we mean the sum of the costs of the interchanges corresponding to its arcs, i.e.,

$$c_\varphi(\tau) = \sum c_\varphi(\alpha_{ij})\{R_{ij} \epsilon \tau\}.$$

Finding a minimal cost tree is of course quite easy. One simply applies the well-known method of KRUSKAL[7] regarding the components of $G\varphi$ as points. The minimal spanning tree in this situation can be assumed to have a special property.

LEMMA 2. *There is a minimal cost spanning tree for G_φ that contains only arcs $R_{i,i+1}$.*

Proof: Suppose τ is a minimal tree containing the arc $R_{ij}, j>i+1$. Since $A_{\varphi(j)}$ and $A_{\varphi(i)}$ are in the same order as B_j and B_i we have

$$\begin{aligned} c_\varphi(\alpha_{ij}) &= \|[B_i,B_j]\cap[A_{\varphi(i)},A_{\varphi(j)}]\| \\ &= \|\{\textstyle\bigcup_{p=i}^{p=j-1}[B_p,B_{p+1}]\}\cap\{\bigcup_{p=i}^{p=j-1}[A_{\varphi(p)},A_{\varphi(p+1)}]\}\| \qquad (12)\\ &\geqq \textstyle\sum_{p=i}^{p=j-1} \|[B_p,B_{p+1}]\cap[A_{\varphi(p)},A_{\varphi(p+1)}]\|. \end{aligned}$$

Because of the order preserving property of φ this is

$$c_\varphi(\alpha_{ij}) \geqq \sum_{p=i}^{p=j-1} c_\varphi(\alpha_{p,p+1}). \qquad (13)$$

The arcs $R_{p,p+1}, p=i, \cdots, j-1$, form a chain linking node i to node j. Therefore if R_{ij} is dropped from τ and these arcs substituted, the graph is still connected, and the total cost of the arcs involved, because of (13), has either decreased or remained the same. By removing superfluous arcs in the new set, one can get down to a new tree τ' of cost $\leqq c_\varphi(\tau)$ and which does not contain R_{ij}.

From now on then we can assume that the minimal spanning trees that are discussed contain only arcs of the form $R_{i,i+1}$.

TREE COSTS AND TOUR COSTS

IF WE start with the minimal permutation φ with cost $c(\varphi)$, and carry out the interchanges of a minimal tree τ in some order, we will get a tour ψ. However the cost $c(\psi)$ will not generally be $c(\varphi)+c_\varphi(\tau)$. This is because, in general, the carrying out of one interchange affects the cost of later interchanges. For example, in Fig. 6, if we take $f(x)=1$ and $g(x)=0$, we have $c_\varphi(\alpha_{12})=\|[1,6]\cap[2,4]\|=2$. But if α_{23} is applied and then α_{12}

$$c_{\varphi\alpha_{23}}(\alpha_{12})=\|[1,6]\cap[2,11]\|=4\neq c_\varphi(\alpha_{12}).$$

However, there are some cases in which no effect is produced on the later interchanges. For instance, in the example just given, one can verify that $c_{\varphi\alpha_{12}}(\alpha_{23})=c_\varphi(\alpha_{23})$. We will discuss these situations in Lemmas 3, 4, and 5.

First, we need two definitions. We will say that the *i*th node is of type 1 relative to a permutation ψ if $B_i\leqq A_{\psi(i)}$, otherwise it is of type 2.

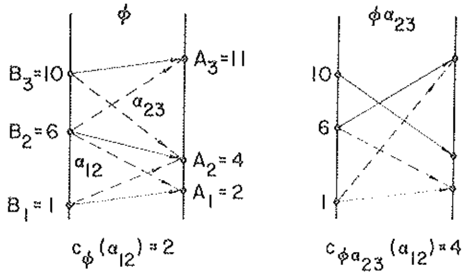

Figure 6

Also, in what follows, we will say a permutation ψ is order preserving on a pair (i,j) if $B_j>B_i\Rightarrow A_{\psi(j)}\geqq A_{\psi(i)}$.

LEMMA 3. *Let ψ be a permutation that is order preserving on (i,j) and (p,q). Let α_{ij} and α_{pq} be interchanges with $i<j$ and $p<q$. Let $\psi'=\psi\alpha_{pq}$. Then $c_\psi(\alpha_{ij})=c_{\psi'}(\alpha_{ij})$ and ψ' is order preserving on (i,j) if any of the following four cases apply:*

(*a*) $p>j$,

(*b*) $q<i$,

(*c*) *$p=j$ and node j is of type 1 relative to ψ,*

(*d*) *$q=i$ and node i is of type 2 relative to ψ.*

Proof. In cases (*a*) and (*b*) $\psi'(i)=\psi(i)$ and $\psi'(j)=\psi(j)$, so the order is unchanged and the intervals entering into the cost formula (11) are also unchanged. In case (*c*) we have $\psi'(i)=\psi(i)$ and $\psi'(j)=\psi(q)$. Order is preserved since $A_{\psi'(j)}=A_{\psi(q)}\geqq A_{\psi(p)}=A_{\psi(j)}\geqq A_{\psi(i)}=A_{\psi'(i)}$. The cost formula (11) is also unchanged since the only difference between $c_\psi(\alpha_{ij})$ and $c_{\psi'}(\alpha_{ij})$ is the replacement of $A_{\psi(j)}$ by $A_{\psi'(j)}$. Since j is assumed to be of type 1, $A_{\psi(j)}$ was already $\geqq B_j$, so replacing it by the even larger $A_{\psi'(j)}$ does not affect the interval intersection that appears in the cost formula. Case (*d*) is similar to (*c*). We have $\psi'(j)=\psi(j)$ and $\psi'(i)=\psi(p)$. $A_{\psi'(i)}=$

$A_{\psi(p)} \leqq A_{\psi(q)} = A_{\psi(i)} \leqq A_{\psi(j)} = A_{\psi'(j)}$, so order is preserved. Again in (11) the only change is the replacement of $A_{\psi(i)}$ by $A_{\psi'(i)} = A_{\psi(p)}$. Since $A_{\psi(p)} \leqq A_{\psi(i)}$ and $A_{\psi(i)} < B_i$, by the type 2 assumption, the intersection in (11) is still unchanged. This established the lemma. Cases (c) and (d) are illustrated in Fig. 7.

We have also established the following property.

LEMMA 4. *In case* (c) *of Lemma* 3, *node* j *is still of type* 1 *relative now to* ψ' *and in case* (d) *node* i *is of type* 2 *relative to* ψ'.

We are now ready for the main lemma.

LEMMA 5. *Let* φ *be the minimal cost permutation,* $\alpha_{i_1 i_1+1}, \alpha_{i_2 i_2+1}, \cdots, \alpha_{i_m i_m+1}$ *a series of interchanges with* $i_1 < i_2 < \cdots < i_m$. *Then there is a* ψ', *obtained by executing the* α*'s in a particular order, with the property that*

$$c(\psi') = c(\varphi) + \sum_{p=1}^{p=m} c_\varphi(\alpha_{i_p i_p+1}).$$

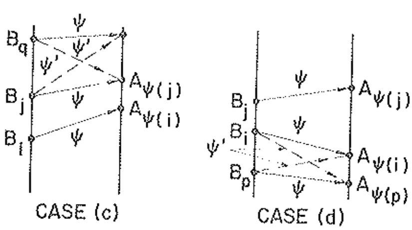

Figure 7

Proof. We will call an interchange a type 1 interchange if its lower node is of type 1; we will call it a type 2 interchange if its lower node is of type 2. The order of execution of the interchanges is as follows. First, do all type 1 interchanges in order of decreasing lower index, i.e., start from the top and work down. Second, do all type 2 interchanges in increasing order of index, i.e., start at the bottom and work up. Consider a type 1 interchange α_{pp+1} encountered during the first part. Let it be the α_{pq} of Lemma 3. Consider the remaining unexecuted interchanges whose cost might be affected by executing α_{pp+1}. If a remaining interchange has no nodes in common with α_{pp+1}, then Lemma 3, case (a) or case (b), applies. If any interchange above α_{pp+1} is unexecuted, it must be of type 2. If it had a node in common with α_{pp+1}, case (d) of Lemma 3 applies. If there is an interchange below α_{pp+1} with a common node, then, since α_{pp+1} is of type 1, Lemma 3, case (c), applies. Consequently, during the first part, the cost of the remaining interchanges is unaffected. It is correct to apply Lemma 3 throughout since the interchanges start out with $B_j > B_i$ implying $A_{\varphi_j} \geqq A_{\varphi_i}$ and Lemma 3 shows that this property is carried forward among the unexecuted interchanges, while Lemma 4 shows that the types of the unexecuted interchanges remain what they were at the beginning.

The reasoning connected with the execution of the type 2 interchanges is similar. Again if a remaining interchange has no node in common with the interchange α_{pp+1} Lemma 3, case (b), applies since the remaining unexecuted interchanges must be above α_{pp+1}. Similarly any interchange with a common node must be above and be of type 2. Then Lemma 3, case (d), applies. This completes the proof.

With the lemmas established, we can now state the main result of this section.

THEOREM 3. *Let τ be a minimal cost spanning tree of G_φ. Let $\alpha^1_{i_1 i_1+1}, \cdots, \alpha^1_{i_l i_l+1}$ be the type 1 interchanges corresponding to the arcs of τ with $i_1 < i_2 < \cdots i_l$. Let $\alpha^2_{j_1 j_1+1}, \cdots, \alpha^2_{j_m j_m+1}$ be the type 2 arcs with $j_1 > j_2 > \cdots j_m$. Then if*

$$\psi^* = \varphi\alpha^1_{i_1 i_1+1} \cdots \alpha^1_{i_l i_l+1}\, \alpha^2_{j_1 j_1+1} \cdots \alpha^2_{j_m j_m+1}$$

ψ^* is a tour with cost

$$c(\psi^*) = c(\varphi) + c_\varphi(\tau).$$

Proof. The fact that ψ^* is a tour comes from Theorem 2, the statement about cost from Lemmas 2 and 4. The order followed in executing the α's is that of the proof of Lemma 5.

ψ^* is of course a candidate for being the minimal cost tour. In the next two sections we prove that it actually is minimal.

AN UNDERESTIMATE OF THE COST OF A PERMUTATION

FOR THE cost of having $\psi(i)$ follow i we have been using (1) which, as in (8), can be written as

$$c_{i\psi(i)} = \|[B_i, +\infty] \cap [-\infty, A_{\psi(i)}]\|_f + \|[-\infty, B_i] \cap [A_{\psi(i)}, +\infty]\|_g. \quad (14)$$

Define the interval P_i, independent of ψ, as

$$P_i = [B_i, B_{i+1}] \cap [A_{\varphi(i)}, A_{\varphi(i+1)}], \quad (15)$$

and the set P, a union of disjoint intervals, as

$$P = \bigcup_{i=1}^{i=N-1} P_i. \quad (16)$$

Using P we define a new cost c^*_{ij} for having j follow i,

$$c^*_{ij} = \|[B_i, +\infty] \cap [-\infty, A_j] \cap P\|_f + \|[-\infty, B_i] \cap [A_j, +\infty] \cap P\|_g, \quad (17)$$

which is the same as (14) except for the presence throughout of P. Using the new cost we can define a new cost $c^*(\psi)$ for permutations by

$$c^*(\psi) = \sum_{i=1}^{i=N} c^*_{i\psi(i)},$$

and for interchanges, just as in the second section, by

$$c_\psi^*(\alpha_{ij}) = c^*(\psi\alpha_{ij}) - c^*(\psi).$$

The formula for $c_\psi^*(\alpha_{ij})$ is obtained in exactly the same way as the formula for $c_\psi(\alpha_{ij})$. The only difference is the presence throughout of P so

$$c_\psi^*(\alpha_{ij}) = \|[B_i,B_j]\cap[A_{\psi(i)},A_{\psi(j)}]\cap P\|, \tag{18}$$

for the case where ψ is order preserving on the pair (i,j). Since $f(x)+g(x)\geqq 0$, we always have $|c_\psi^*(\alpha_{ij})| \leqq |c_\psi(\alpha_{ij})|$, and in the case where (18) applies we have

$$c_\psi^*(\alpha_{ij}) \leqq c_\psi(\alpha_{ij}). \tag{19}$$

We can now prove the theorem that gives the underestimate.

THEOREM 4. *For any permutation* ψ,

$$c(\psi) \geqq c(\varphi) + c^*(\psi).$$

Proof. In this proof and in what follows we will want to refer to the contribution to the cost $c^*_{i\psi(i)}$ made by a single interval P_q. So we define $c^*_{ij}(q)$ by

$$c^*_{ij}(q) = |[B_i,+\infty]\cap[-\infty,A_j]\cap P_q|_f + |[-\infty,B_i]\cap[A_j,+\infty]\cap P_q|_g, \tag{20}$$

which is merely (17) with P_q replacing P. Clearly

$$\textstyle\sum_{q=1}^{q=N-1} c^*_{i\psi(i)}(q) = c^*_{i\psi(i)}.$$

Let us first establish Theorem 4 for the case $\psi=\varphi$. We will do this by showing that $c^*(\varphi)=0$.

For this purpose, and a later one, we prove the following lemma:

LEMMA 6. *For any i,j and q, whenever the intervals*

$$[B_i,+\infty]\cap[-\infty,A_{\psi(i)}]\cap P_q, \tag{21a}$$

$$[A_{\psi(j)},+\infty]\cap[-\infty,B_j]\cap P_q, \tag{21b}$$

are nonempty they are P_q, and should P_q be nonempty, they will be P_q if and only if

$$i \leqq q < \varphi^{-1}\psi(i) \tag{22a}$$

$$\varphi^{-1}\psi(j) \leqq q < j \tag{22b}$$

hold respectively.

Proof. Since P_q is $[B_q,B_{q+1}]\cap[A_{\varphi(q)},A_{\varphi(q+1)}]$ the interval (21a) would be empty if $i>q$ since then the interval $[B_i,+\infty]\cap[B_q,B_{q+1}]$ would be empty, and also if $q\geqq\varphi^{-1}\psi(i)$ since then the interval $[-\infty,A_{\psi(i)}]\cap[A_{\varphi(q)},A_{\varphi(q+1)}]$ would be empty. Should (22a) hold then for (21a) to be other than P_q one of B_i and $A_{\psi(i)}$ would have to lie interior to both the intervals $[B_q,B_{q+1}]$ and $[A_{\varphi(q)},A_{\varphi(q+1)}]$, which is impossible by the indexing of the B's and by the definition of φ. A similar argument can be repeated for (21b) and (22b).

Applying the lemma to the case $\psi=\varphi$, neither (22a) or (22b) can be satisfied so that the intervals (21a) and (21b) are always empty. Consequently $c^*_{i\varphi(i)}(q)=0$ for all i and q and therefore $c^*(\varphi)=0$. We define the 'height' h of a permutation ψ as the first index on which $\psi(i)$ differs from $\varphi(i)$. If $h=N,\psi=\varphi$ and the theorem holds. Let us assume the theorem holds for all permutations of heights $h,h\geqq n$, and prove that it then holds for all permutations of height $n-1$.

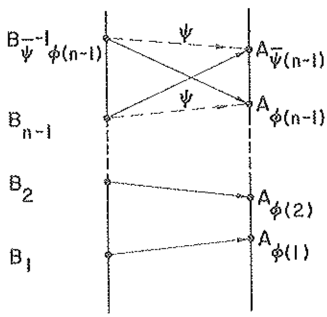

Figure 8

Let $\bar{\psi}$ have height $n-1$ and let ψ be defined by

$$\psi=\bar{\psi}\alpha_{ij},$$

with $i=n-1,j=\bar{\psi}^{-1}\varphi(n-1)$. (*See* Fig. 8 where the solid arrows are for $\bar{\psi}$.) Then

$$\psi(n-1)=\bar{\psi}[\bar{\psi}^{-1}\varphi(n-1)]=\varphi(n-1)$$

so that ψ is of height n and the theorem holds for ψ. Of course $\bar{\psi}$ is obtained from ψ by the same interchange α_{ij}. Since $i<j$ and $\varphi^{-1}\psi(i)=n-1<\varphi^{-1}\psi(j)=\varphi^{-1}\bar{\psi}(n-1),\psi$ is order preserving on the pair (i,j) and therefore the interchange formulas (11) and (18) hold without a minus sign. From the induction hypothesis

$$c(\psi)\geqq c(\varphi)+c^*(\psi),$$

while from (19) $$c_\psi(\alpha_{ij})\geqq c^*_\psi(\alpha_{ij}),$$

so that adding we find

$$c(\bar{\psi})=c(\psi)+c_\psi(\alpha_{ij})\geqq c(\varphi)+c^*(\psi)+c^*_\psi(\alpha_{ij})=c(\varphi)+c^*(\bar{\psi}),$$

which established the theorem.

AN UNDERESTIMATE OF THE COST OF A TOUR

WE MUST now make a connection between our cost underestimate and the cost of the minimal spanning tree.

Let us define the n-node graph $G_\psi{}^*$. $G_\psi{}^*$ contains all the arcs $R_{q\varphi(q)}$ of G_φ as well as any undirected arc $R_{q,q+1}$ for every q for which either for some i (22a) holds, or for some index j (22b) holds. We will show later in Lemma 8 that these latter arcs R_{qq+1} of $G_\psi{}^*$ are just those for which the interval P_q contributes both to the f cost and the g cost in $c^*(\psi)$.

The next lemma implies that $G_\psi{}^*$ contains a spanning tree.

LEMMA 7. *If ψ is a tour $G_\psi{}^*$ is connected.*

Proof. Suppose $G_\psi{}^*$ is not connected. Then it is possible to divide it into two disjoint components C_1 and C_2. Since $G_\psi{}^*$ includes all the arcs of G_φ, C_1, and C_2 can only be unions of components of G_φ. Therefore if $\bar{C}_1$ is the set of nodes in C_1

$$\varphi\bar{C}_1=\bar{C}_1.$$

Now suppose that

$$\varphi^{-1}\psi\bar{C}_1=\bar{C}_1, \tag{24}$$

then multiplying by φ gives

$$\psi\bar{C}_1=\varphi\bar{C}_1=\bar{C}_1.$$

Since this would imply that ψ is not a tour, (24) cannot hold and there must be at least one i_0 such that the node i_0 is in C_1 and the node $\varphi^{-1}\psi(i_0)$ is in C_2. If $i_0<\varphi^{-1}\psi(i_0)$, let q be the smallest index for which

$$i_0\leqq q<q+1\leqq\varphi^{-1}\psi(i_0),$$

and for which the node $q+1$ is in C_2. By definition, the arc $R_{q,q+1}$ is in $G_\psi{}^*$ in this case.

If $\varphi^{-1}\psi(i_0)<i_0$, let q be the largest index for which $\varphi^{-1}\psi(i_0)\leqq q<q+1\leqq i_0$, and for which the node q is in C_2. Again, by definition, the arc R_{qq+1} is in $G_\psi{}^*$ in this case too.

But R_{qq+1} links the supposedly disjoint components C_1 and C_2. This contradiction establishes the lemma.

Before proving the minimality of the tour ψ^* of Theorem 3 we need two further lemmas:

LEMMA 8. *Given a fixed q and a permutation ψ, the number of values of the index i for which (22a) holds is the same as the number of values of j for which (22b) holds.*

Proof. Define the function $\bar{R}(i)$ by

$$\bar{R}(i)=1 \quad \text{if} \quad i>q,$$

$$\bar{R}(i)=0 \quad \text{if} \quad i\leqq q.$$

Then for any permutation ψ

$$\textstyle\sum_i \{\tilde{R}(i)-\tilde{R}[\varphi^{-1}\psi(i)]\}=0, \tag{23}$$

since $\varphi^{-1}\psi$ merely rearranges the terms. The individual terms

$$\{\tilde{R}(i)-\tilde{R}[\varphi^{-1}\psi(i)]\}$$

will be $+1$ if (22b) holds, -1 if (22a) holds, and zero otherwise. (23) shows that the number of $+1$ and -1 terms must be equal, which establishes Lemma 8.

We are now in a position to prove the minimality of ψ^*.

Consider the sum $c^*(\psi)$, which can be written:

$$c^*(\psi)=\textstyle\sum_i c^*_{i\psi(i)}=\sum_q\{\sum_i c^*_{i\psi(i)}(q)\}. \tag{25}$$

Let us look more closely at the term in parentheses

$$\textstyle\sum_i c^*_{i\psi(i)}(q). \tag{26}$$

By Lemma 6, if $c^*_{i\psi(i)}(q)$ is not zero then it is either $|P_q|_f$ or $|P_q|_g$; the former when (22a) holds and the latter when (22b) holds for $j=i$. By Lemma 8 we can conclude that the $|P_q|_f$ and $|P_q|_g$ occur equally often in the sum (26) so that that sum is always a nonnegative integer multiple of $\|P_q\|$ and therefore itself nonnegative. Should R_{qq+1} be an arc of $G_\psi{}^*$ then necessarily (22a) or (22b) holds for some i or j and therefore by Lemma 8 (22a) holds for some i and (22b) for some j. Consequently again from Lemma 6 we can conclude that

$$\textstyle\sum_i c^*_{i\psi(i)}(q)\geqq\|P_q\|.$$

So from (25)

$$c^*(\psi)\geqq\textstyle\sum_q\|P_q\|\ \{q|R_{qq+1}\epsilon G_\psi{}^*\},$$

and since $\|P_q\|=c_\varphi(\alpha_{qq+1})$

$$c^*(\psi)\geqq\textstyle\sum_q c_\varphi(\alpha_{qq+1})\ \{q|R_{qq+1}\epsilon G_\psi{}^*\}.$$

Since $G_\psi{}^*$ is connected, it includes some tree τ'. So

$$\begin{aligned} c^*(\psi)&\geqq\textstyle\sum_q c_\varphi(\alpha_{qq+1})\ \{q|R_{qq+1}\epsilon G_\psi{}^*\}\\ &\geqq\ \textstyle\sum q\, c_\varphi(\alpha_{qq+1})\ \{q|R_{qq+1}\epsilon\tau'\}\geqq c_\varphi(\tau'). \end{aligned} \tag{27}$$

If τ is any minimal spanning tree we have from (27)

$$c^*(\psi)\geqq c_\varphi(\tau). \tag{28}$$

Now from Theorem 4, (28) above, and Theorem 3,

$$c(\psi)\geqq c(\varphi)+c^*(\psi)\geqq c(\varphi)+c_\varphi(\tau)=c(\psi^*), \tag{29}$$

where the ψ^* of (29) is of course the candidate for an optimal tour described in Theorem 3.† (29) establishes Theorem 5.

THEOREM 5. *The tour ψ^* described in Theorem 3 is a minimal cost tour.*

THE BOTTLENECK CASE

ANOTHER objective function that can be considered is $m(\psi)$ defined for any permutation as $\max_{1 \leqq i \leqq n}\{c_{i\psi(i)}\}$. The problem of minimizing $m(\psi)$ for any permutation ψ is the bottleneck assignment problem already solved by Gross in reference 4. The problem of minimizing $m(\psi)$ for tour ψ is the bottleneck traveling salesman problem which we take up here for the cost matrix $\{c_{ij}\}$ defined by (1) when $f(x) \geqq 0$ and $g(x) = 0$. The results obtained here apply equally well to the case when $f(x) = 0$ and $g(x) \geqq 0$. We do not yet know how to solve the problem for general f and g.

The method for obtaining a tour ψ' to minimize $m(\psi')$ is but slightly different from the method to obtain one minimizing $c(\psi')$. The difference lies in the costs assigned to the interchanges used to obtain a minimum spanning tree of G_φ, and in the order in which the interchanges are applied to φ. Instead of the cost $c_\varphi(\alpha_{qq+1})$ an arc R_{qq+1} of a spanning tree of G_φ has a cost $c_{q\varphi(q+1)}$. Having obtained a minimum spanning tree of G_φ with these new costs and with arcs say $R_{j_1 j_1+1}, \cdots, R_{j_m j_m+1}$ then ψ' is defined:

$$\psi' = \varphi\alpha_{j_1 j_1+1} \cdots \alpha_{j_m j_m+1},$$

where $j_1, \cdots, j_m$ are in increasing order of size. Some preliminary results are necessary to show that ψ' does actually minimize $m(\psi')$.

Let ψ be any tour. Consider the graph G_ψ^* previously defined. We will show that if R_{qq+1} is an arc of G_ψ^* not in G_φ then

$$m(\psi) \geqq c_{q\varphi(q+1)}. \tag{30}$$

Whenever R_{qq+1} is such an arc then either there exists an i for which (22a) holds or a j for which (22b) holds; but we have seen from Lemma 8 that should there exist such a j then there also exists such an i. Hence necessarily for some i (22a) holds. For this i then

$$[B_q, +\infty] \cap [-\infty, A_{\varphi(q+1)}] \subseteq [B_i, +\infty] \cap [-\infty, A_{\psi(i)}]$$

and consequently since

$$m(\psi) \geqq c_{i\psi(i)}$$

(30) holds. It follows therefore that any tour ψ defines a spanning tree of G_φ with an arc of greatest cost not exceeding $m(\psi)$.

† In the fourth section we proved Lemma 2 to show that only the arcs R_{qq+1} need be used for the arcs of a spanning tree of G_φ. The result we have just established indicates that we could have done without this lemma. However it does motivate early in the proof the limitation to spanning trees with only the arcs R_{qq+1}.

From the manner in which ψ' has been defined it is evident that for some j either $m(\psi')=c_{j\varphi(j)}$ or $m(\psi')=c_{j\varphi(j+1)}$. In the latter case $m(\psi')$ must be identical with the largest cost attached to an arc of the minimum spanning tree of G_φ and consequently for any tour $\psi, m(\psi)\geqq m(\psi')$. Assume therefore that $m(\psi')=c_{j\varphi(j)}$ and that for some tour ψ $m(\psi)<m(\psi')$. Then necessarily $\varphi^{-1}\psi(j)<j$ and therefore if $q=j-1$, it follows that

$$\varphi^{-1}\psi(j)\leqq q<j.$$

From Lemma 8 again there exists an i such that $i\leqq q<\varphi^{-1}\psi(i)$ and consequently $i<j\leqq\varphi^{-1}\psi(i)$. But then

$$m(\psi)\geqq c_{i\psi(i)}=\|[B_i, +\infty]\cap[-\infty, A_{\psi(i)}]\|_f$$
$$\geqq\|[B_j, +\infty]\cap[-\infty, A_{\varphi(j)}]\|_f=c_{j\varphi(j)}=m(\psi'),$$

contradicting $m(\psi)<m(\psi')$.

Incidentally this latter argument can be applied to prove that φ is a permutation minimizing $m(\varphi)$.

In the final section we will give an example to show that in general the tour minimizing the bottleneck objective is different from the tour minimizing the sum objective. For one special case, however, these tours will be identical; that is when $B_i\geqq A_{\varphi(i)}$ for all i. For in this case

$$c_\varphi(\alpha_{qq+1})=\|[B_q, B_{q+1}]\cap[A_{\varphi(q)}, A_{\varphi(q+1)}]\|_f$$
$$=\|[B_q, +\infty]\cap[-\infty, A_{\varphi(q+1)}]\|_f$$
$$=c_{q\varphi(q+1)},$$

and hence the costs attached to the arcs of a spanning tree of G_φ are the same in the two cases. Furthermore all the interchanges for a spanning tree of G_φ will be of type 2 so that the order in which they are applied to φ to yield the ψ^* of Theorem 3 is the same as the order described for obtaining ψ'.

These results permit us to solve a well-known problem of graph theory for a special class of graphs. The problem is that of characterizing those directed graphs possessing Hamiltonian circuits; *see* for example Section 3.4 of reference 9.

For a node i of a directed graph G let Γ_i be the set of nodes j for which an arc of G is directed from i to j.

THEOREM 6. *Let G be a directed graph of n nodes for which $\Gamma_1\subseteqq\Gamma_2\subseteqq\cdots\subseteqq\Gamma_n$, and $\Gamma_n=\{1, 2, \cdots, n\}$; then necessarily there will be a permutation φ with the property that for each i there is a j for which $\Gamma_i=\{\varphi(1), \cdots, \varphi(j)\}$. Let G_φ' be an undirected graph of n nodes with an arc $R_{q\varphi(q)}$ if and only if $\varphi(q)\in\Gamma_q$ and an arc R_{qq+1} if and only if $\varphi(q+1)\in\Gamma_q$. Then a necessary and sufficient condition that G possess a Hamiltonian circuit is that G_φ' be connected.*

Proof. Without loss it is possible to assume that $\Gamma_1 \neq 0$ since without this assumption neither can G possess a Hamiltonian circuit nor can G_φ' be connected.

For each i define $A_i = \varphi^{-1}(i)$ and define $B_i = |\Gamma_i|$ so that $B_1 \leqq B_2 \leqq \cdots \leqq B_n$ and $A_{\varphi(1)} \leqq A_{\varphi(2)} \leqq \cdots \leqq A_{\varphi(n)}$ as previously. Further let $f(x) = 1$ and $g(x) = 0$ so that for any i and j, $c_{ij} = 0$ if and only if $B_i \geqq A_j$, or equivalently, $|\Gamma_i| \geqq \varphi^{-1}(j)$. But $|\Gamma_i| \geqq \varphi^{-1}(j)$ if and only if

$$\Gamma_i \supseteq \{\varphi(1), \varphi(2), \cdots, \varphi[\varphi^{-1}(j)]\},$$

so that consequently $c_{ij} = 0$ if and only if $j \epsilon \Gamma_i$. For any G satisfying the hypothesis of the theorem it follows that ψ is a Hamiltonian circuit if and only if ψ is a tour for which $m(\psi) = 0$. But now applying the results of this section we have that there exists a Hamiltonian circuit for G if and only if there exists a spanning tree of G_φ each arc R_{qq+1} of which has a cost $c_{q\varphi(q+1)}$ which is zero; that is, if and only if G_φ' is connected.

THE ALGORITHM, EXAMPLES AND REMARKS

Description of the Algorithm

We start by giving a description of the computational steps required by the minimal tour algorithm, which have been justified in the preceding sections.

The data of the problem are the functions $f(x)$ and $g(x)$, which give the cost of marginally increasing or decreasing the state, and a list of N jobs J_i with their starting state values A_i and ending state values B_i.

We go through the following preliminary steps.

P1. Arrange the numbers B_i in order of size and renumber the jobs so that with the new numbering

$$B_i \leqq B_{i+1}, \qquad i = 1, \cdots, N-1.$$

P2. Arrange the A_i in order of size.

P3. Find $\varphi(p)$ for all p. The permutation φ is defined by

$$\varphi(p) = q,$$

q being such that A_q is the pth smallest of the A_i.

P4. Compute the numbers $c_\varphi(\alpha_{ii+1})$ for $i = 1, \cdots, N-1$. $c_\varphi(\alpha_{ii+1})$ is defined by

$$c_\varphi(\alpha_{ii+1}) = 0 \qquad \text{if } \max(B_i, A_{\varphi(i)}) \geqq \min(B_{i+1}, A_{\varphi(i+1)}),$$

$$c_\varphi(\alpha_{ii+1}) = \int_{\max(B_i, A_{\varphi(i)})}^{\min(B_{i+1}, A_{\varphi(i+1)})} \{f(x) + g(x)\}\, dx$$

$$\text{if } \max(B_i, A_{\varphi(i)}) < \min(B_{i+1}, A_{\varphi(i+1)}).$$

TABLE I

B	*A*
31	34
19	45
3	4
40	18
26	22
15	16
1	7

TABLE II

No.	*B*	*A*	No.
7	40	45	4
6	31	34	6
5	26	22	5
4	19	18	7
3	15	16	3
2	3	7	1
1	1	4	2

We now select a set of arcs by the following steps.

S1. Form an undirected graph with N nodes and undirected arcs connecting the ith and φth nodes $i=1, \cdots, N$.

S2. If the current graph has only one component, go to step T1. Otherwise select the smallest value $c_\varphi(\alpha_{i,i+1})$ such that i is in one component and $i+1$ in another. In case of a tie for smallest, choose any.

S3. Adjoin the undirected arc $R_{i,i+1}$ to the graph using the i value selected in S2. Return to S2.

T1. Divide the arcs added in S3 into two groups. Those $R_{i,i+1}$ for which $A_{\varphi(i)} \geqq B_i$ go in group 1, those for which $B_i > A_{\varphi(i)}$ go in group 2.

T2. Find the largest index i_1 such that R_{i_1,i_1+1} is in group 1. Find the second largest i_2, etc., up to i_l, assuming there are l elements in group 1.

T3. Find the smallest index j_1 such that R_{j_1,j_1+1} is in group 2. Find the second smallest j_2, etc., up to j_m, assuming there are m elements in group 2.

T4. The minimal tour is obtained by following the ith job by the job $\psi^*(i)$,

$$\psi^*(i) = \varphi\alpha_{i_1,i_1+1}\alpha_{i_2,i_2+1}\cdots\alpha_{i_l,i_l+1}\alpha_{j_1,j_1+1}\alpha_{j_2,j_2+1}\cdots\alpha_{j_m,j_m+1}(i).$$

In the above expression the permutation α_{pq} is defined to be

$$\begin{cases}\alpha_{pq}(p)=q,\\ \alpha_{pq}(q)=p,\\ \alpha_{pq}(i)=i, & (i\neq p,\ q)\end{cases}$$

TABLE III

i	$\psi^*(i)$
1	2
2	7
3	1
4	5
5	6
6	3
7	4

and the order of applying the permutations is from right to left in the sequence given.

We next show these steps in a numerical example.

Numerical Example

For our numerical example we take a problem with $f(x)=1$, and $g(x)=0$. There are seven jobs with A's and B's as given in Table I.

In step (P1) we rank and number the jobs to obtain the first two columns of Table II that give the job number or rank i, and the value of B_i.

In step (P2) we arrange the A_i that appear in column 3 of Table II with their job numbers next to them in column 4.

The permutation φ of step (P3) is given by the first and fourth columns of Table II; the entry in the ith row of column 4 is $\varphi(i)$.

In carrying out the next step, (P4), we need various numbers such as max $(B_i, A_{\varphi(i)})$. max $(B_i, A_{\varphi(i)})$ is the larger of the two numbers appearing in the ith row of Table II in columns 2 and 3. Computing the $c_\varphi(\alpha_{ii+1})$ we find

$$c_\varphi(\alpha_{1,2})=0,$$

$$c_\varphi(\alpha_{2,3})=\int_7^{15} dx=8,$$

$$c_\varphi(\alpha_{3,4})=\int_{16}^{18} dx=2,$$

$$c_\varphi(\alpha_{4,5})=\int_{19}^{22} dx=3,$$

$$c_\varphi(\alpha_{5,6})=\int_{26}^{31} dx=5,$$

$$c_\varphi(\alpha_{6,7})=\int_{34}^{40} dx=6.$$

We are now ready for (S1). We form the graph of Fig. (9) using the arcs $R_{i\varphi(i)}$ that are, aside from arcs of the form R_{ii} that do no connecting, are $R_{7,4}$ and $R_{2,1}$.

To carry out (S2) and (S3) it is convenient to adjoin to the graph of Fig. (9) dotted lines for arcs R_{ii+1} together with their costs, Fig. (10a).

In carrying out (S2) and (S3) repeatedly we choose in succession $R_{3,4}$, $R_{4,5}$, $R_{5,6}$, and R_{23}, which gives a connected graph, Fig. (10b).

To carry out (T1) we divide the arcs into two groups. Group 1, $\{R_{2,3}, R_{3,4}\}$, Group 2, $\{R_{4,5}, R_{5,6}\}$.

(T2) gives

$$i_1=3,$$

$$i_2=2,$$

and (T3) gives

$$j_1=4,$$

$$j_2=5.$$

Then (T4), the minimal tour, is given by the permutation

$$\psi^*=\varphi\alpha_{3,4}\alpha_{2,3}\alpha_{4,5}\alpha_{5,6},$$

Figure 9

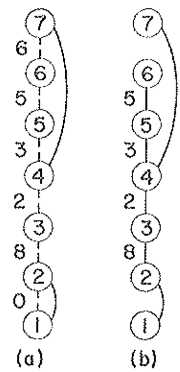

Figure 10

whose values appear in Table III. Since $\psi^*(i)$ gives the successor to the ith job the minimal cost tour is 1-2-7-4-5-6-3-1. Its cost is 34. This may be computed directly, using the changeover costs, or by computing the cost of φ, which is 16, and then adding the costs of the interchanges used, which adds 18.

Numerical Example in the Bottleneck Case

In the bottleneck case the costs $c_{i\varphi(i+1)}$ must be computed. They are:

$$c_{1\varphi(2)}=7-1=6,$$

$$c_{2\varphi(3)}=16-3=13,$$

$$c_{3\varphi(4)}=18-15=3,$$

$$c_{4\varphi(5)}=22-19=3,$$

$$c_{5\varphi(6)}=34-26=8,$$

$$c_{6\varphi(7)}=45-31=14.$$

In place of the graph of Fig. 10a we now have Fig. 11.

The minimum spanning tree for this graph has exactly the same edges as the one in Fig. 10b even though the costs are different. But the interchanges corresponding to the edges of the tree are applied to φ in a different order; the minimal tour ψ' is defined:

$$\psi' = \varphi\alpha_{2,3}\alpha_{3,4}\alpha_{4,5}\alpha_{5,6},$$

and is 1-2-3-7-4-5-6-1.

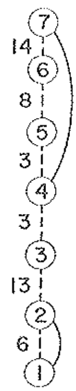

Figure 11

Sequences and Tours

Having worked out a tour example we return to the question of the connection between the sequencing problem and the tour problem. In the introduction we observed that the problems are the same provided that the sequencing problem has specified initial and final states. If there is no such requirement there is still a connection if either $g(x)$ or $f(x)$ is zero. Let us now assume that $g(x)=0$.

Case (i). An initial state B_0 is given, but the final state can be anything. In this case the sequencing problem is equivalent to the tour problem with an additional job J_0, if J_0 has starting state $A_0 \leqq \min_{i=1,\ldots,N} B_i$ and ending state B_0.

Case (ii). Any state is available at the beginning but the final state must be A_0. Then the equivalent tour involves the additional job J_0 with starting state A_0 and ending state $B_0 \geqq \max_{i=1,\ldots,N} A_i$.

Case (iii). Any state is available at the beginning and the machine can be left in any state. Then the equivalent tour has a J_0 with

$$A_0 \leqq \min_{i=1,\ldots,N} B_i$$

and $B_0 \geqq \max_{i=1,\ldots,N} A_i$.

The reasoning in all these cases is the same as that of the introduction.

We need only show in each case that the cost of the sequence equals the cost of the corresponding tour. This is so in *Case* (*i*) because, by using our tour, we force the machine to end its sequence in state A_0; it can switch into A_0 from any possible ending state B_i at zero cost because $A_0 \geqq B_i$ and $g(x)=0$.

The same argument applies in *Cases* (*ii*) and (*iii*) and also when $g(x)\geqq 0$ and $f(x)=0$.

One-Sided Cost Functions

Because of the special role played by the cases where one of the cost functions is zero, we state the following theorem.

THEOREM 7. *Let ψ^* be a minimal cost tour in a problem with cost functions $f(x)$ and $g(x)$. Then ψ^* is still a minimal cost tour in the new problem obtained by replacing $f(x)$ by $f(x)+g(x)$ and $g(x)$ by 0.*

Proof. In carrying out the algorithm ψ^* is determined from φ, which, because it depends only on the A_i and B_i, is the same in both problems, and from the $c_\varphi(\alpha_{ii+1})$, which, because they depend on the sum of the two cost functions, are the same in both problems. This ends the proof.

The equivalence between the f, g problem and the $f+g$, 0 problem is not complete however. The cost $c(\psi^*)$ is not the same in the two problems because $c(\varphi)$ can change, and the equivalence does not extend to the free ended sequencing problems such as (*i*), (*ii*), and (*iii*) above. More precisely a problem such as (*i*), (*ii*), or (*iii*), but with cost functions $f(x)$ and $g(x)$ both nonzero, does not in general have the same optimal sequence as the same problem with cost functions $f(x)+g(x)$ and 0.

Further Examples

Turning now to further examples, we illustrate the case $f(x)\geqq 0$, $g(x)=0$ by an application closely related to a sequencing problem discussed in reference 3.

A pair of axles with slitting knives on them are being used one at a time to cut an endless strip of cardboard as it runs beneath the knives. Various arrangements of the knives are needed. Each arrangement takes a time A_i to prepare and will be used in cutting for a time B_i. While one arrangement is being used for cutting, the knives in the other axle are being positioned for the next arrangement. If J_i precedes J_j and $B_i \geqq A_j$, then the arrangement for J_j can be completed during the run of J_i. Otherwise the machine is stopped and there is a delay of $A_j - B_i$ while the arrangement for J_j is completed. The cost involved is the delay, so $f(x)=1$, and $g(x)=0$. The state of the machine is the amount of time since the start of the last job. The sequencing problem of reference 3 differs from this problem in having three axles rather than two.

A quite different sounding example of a one state-variable machine is a truck running along a road. The truck picks up a load at point A_i and delivers it at B_i; it carries only one load at a time. The time spent traveling from delivery points to pick-up points is wasted and is to be minimized. The position of the truck on the road is the state variable, and the speeds the truck can attain on the various parts of the road in one direction or the other provide the $f(x)$ and $g(x)$.

Relations to the Two-Machine Sequencing Problem

Finally we take up the relation between the problem discussed here and the well-known two-machine sequencing problem solved by S. M. Johnson.[6] We will discuss this relation from various points of view.

First, Johnson's problem can be regarded as being, in some sense, a one state-variable machine problem. The state in which the machine is left after a job is the difference in the time of completion of that job on the second and first machines. If this is X_i, then the next job accepts any starting state and leaves a final state X_{i+1} given by

$$X_{i+1} = B_{i+1} + \max(0, X - A_{i+1}),$$

where A_{i+1} is the time spent on the first machine and B_{i+1} the time spent on the second. Thus there is a state transformed by the A's and B's but in a manner different from the one we have discussed.

Secondly, one can also see that the cutting-knife example we discussed suggests the Johnson problem. Setting up the back-up axle is equivalent to spending time A on machine 1, and the running time B is equivalent to spending time on machine 2. However, in our example, even during a very long run, only one arrangement can be set up on the reserve axle because there is only one reserve axle. In Johnson's problem, a great many jobs could be completed on the first machine while the second machine was doing a long run on one job. Johnson's problem then can be seen to be a version of our cutting problem but with an unlimited array of reserve axles.

Finally, one may wonder if, despite the apparent differences, Johnson's method, applied to the A's and B's, might yield the same answer as ours: perhaps from some other viewpoint, the apparent differences would disappear or cancel out. That this cannot happen is shown by the following. Johnson's optimal sequence depends only on the ranking of the assembled numbers A_i and B_i. Any change in these numbers that preserves ranking does not change the choice of optimal sequence. However, these changes do affect the $c_\varphi(\alpha_{i,i+1})$ of this paper and can and do change therefore the choice of optimal sequence. So the answers, in general, cannot be the same.

REFERENCES

1. R. L. Ackoff (ed.), *Progress in Operations Research*, Vol. I, Wiley, New York, 1961.
2. P. C. Gilmore and R. E. Gomory, "A Solvable Case of the Travelling Salesman Problem," *Proc. Nat. Acad. Sci.* **51**, 178–181 (1964).
3. ——— and ———, "Multi-Stage Cutting Stock Problems of Two and More Dimensions," IBM Research Report RC 1099, Jan, 20, 1964.
4. O. Gross, "The Bottleneck Assignment Problem," The Rand Corp. P-1630, March 6, 1959.
5. Michael Held and Richard M. Karp, "A Dynamic Programming Approach to Sequencing Problems." *J. Soc. Indust. and Appl. Math.* **10**, 196–210 (1962).
6. S. M. Johnson, "Optimal Two- and Three-Stage Production Scheduling with Setup Times Included," *Naval Res. Log. Quart.*, **1**, 61–68 (1954).
7. J. B. Kruskal, "On the Shortest Spanning Subtree of a Graph and the Travelling Salesman Problem," *J. Soc. Indust. and Appl. Math.* **5**, 32–38 (1957).
8. John D. C. Little, Katta G. Murty, Dura W. Sweeney, and Caroline Karel, "An Algorithm for the Travelling Salesman Problem," *Opns. Res.* **11**, 972–989 (1964).
9. Oystein Ore, "Theory of Graphs," Am. Math. Soc. Colloquium Publ. **38**, 1962