

# A Ricardo model with economies of scale

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**ABSTRACT** A model of international trade that resembles the classical Ricardo model but differs from it in admitting economies of scale in production is described.

This note describes a model of international trade that resembles the classical Ricardo model but differs from it in admitting economies of scale in production.

I mention with pleasure the many contributions of Herbert E. Scarf without which this paper would not have been written.

Allowing economies of scale does of course have a profound effect on the behavior of the model. One aspect of almost any economy of scale is that it gives an advantage to countries that are actually engaged in production of a given good, as opposed to those who are not participating at all. This barrier to entry tends to stabilize the production status quo, whatever it is, and leads to a multitude of possible equilibrium points. These different equilibrium points represent vastly different outcomes for the countries involved, either in terms of national income or in terms of utility.

Equilibrium points in this model are virtually the same as in the ordinary Walras equilibrium model. At each of our many equilibrium points there are prices and wages at which supply equals demand for each good. The wage bill for each producer equals the value of goods produced, while for nonproducers the profit for entering into production at these wages and prices, and for low levels of production, is negative or zero. While in the presence of a linear model or of diseconomies of scale these conditions would provide a single equilibrium, they lead here, inherently, to many. The situation mirrors the unavoidable differences between local optimization with convexity and local optimization in the presence of nonconvexity.

Integer variables enter naturally into this model through a set of 0–1 variables that determine which country is to be a producer of a given good and which is not. Finding production patterns whose associated equilibrium points maximize utility then becomes an integer programming (maximization) problem.

Unlike the classical Ricardo model, the most efficient producer will not always be the one who produces in this model. An entrenched economy of scale can be a barrier that prevents effective competition from a nonproducer, even one with a superior production function. However, Ricardo-like concepts can be reintroduced into the model with the concepts of Ricardo level and Ricardo solution. The Ricardo solution is one in which goods are produced only by the most efficient producer, while the Ricardo level is the exchange rate at which this is possible. There is always a Ricardo level but not always a Ricardo solution in the presence of economies of scale.

A typical two-country outcome is illustrated in Fig. 1, which is based on the data of Table 1. Fig. 1 plots Cobb–Douglas utility vs. a normalized national income  $Y$  for

country 1. Each dot in the figure is an equilibrium point. The large dots are outcomes in which only one country is a producer for each good. The exchange rate ratio  $w_1/w_2$  corresponding to the national income is plotted on the top horizontal line, the Ricardo level is the vertical bar, and the utility in autarchy is marked by the horizontal bar on the right. This example has eight products (or eight industries.)

There are several aspects of Fig. 1 worth noting. First the outcomes form an array of points with a definite and characteristic shape, equilibrium points are not just anywhere. This shape recurs throughout our limited empirical experience, and a rough rationale for it will be given. Second, the upper edge of the array of outcomes is rather well defined; in Fig. 1 it is marked by a dotted line. The equilibria near this boundary are the ones that maximize utility. I will show that this boundary line can be computed by a separate and rapid calculation without computing the assemblage of equilibrium points. Third, utility does not increase indefinitely with national income but rather decreases after a certain point.

Fig. 2 shows the utility of country 1 on the right vertical axis and the utility of country 2 on the left. The horizontal axis is  $Y$ , and in this normalization the national income of country 2 is  $1 - Y$ . There are two dots for each equilibrium point, giving the two utilities  $U_1$  and  $U_2$ . Each utility is normalized so that its greatest utility is 1. Both autarchy levels are shown. In this example there is a Ricardo point, and it is marked by the two black squares.

## Existence of Solutions

The classical Ricardo model makes the assumption that production exhibits constant returns to scale. This assumption rules out the possibility of set-up costs in the initiation of manufacturing activities, as well as other production functions having economies of scale. In this model the production functions  $f_{i,j}$  do have economies of scale.  $f_{i,j}(l_{i,j})/l_{i,j}$  will always be a nondecreasing function of the labor input  $l_{i,j}$ . Here the index  $i$  denotes the  $i$ th good and  $j$  the  $j$ th country. We shall limit the number of trading partners to be two as in the Ricardo model.

The Cobb–Douglas utility will be used throughout, so for country  $j$

$$u_j(q_1, \dots, q_m) = \sum_i d_{i,j} \ln q_i, \quad d_j > 0, \quad \sum_i d_{i,j} = 1,$$

with  $q_i$  the quantity of the  $i$ th good. This implies that country  $j$  spends a constant fraction  $d_{i,j}$  of its national income  $N_j$  on good  $i$ , for all prices.

In the classical model, the pattern of specialization is determined by the equilibrium solution. In this formulation with increasing returns to scale, we will have to deal with many equilibrium solutions and the possibility of essentially arbitrary choices of the manufacturing industries being active in each country.

For each  $j$  let  $S(j)$  be the set of commodities produced in country  $j$ . Given such a pattern of specialization, an average-cost pricing equilibrium is a price vector  $\pi_i$  a set of wage rates  $w_j$ , and an allocation  $l_{i,j}$  of each country's labor supply  $L_j$

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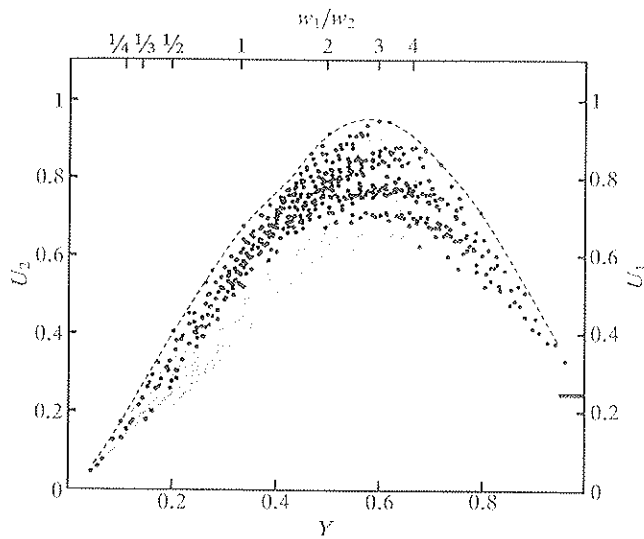


FIG. 1. Cobb-Douglas utility vs. normalized national income  $Y$  for country 1.

among those industries in which it specializes such that the following conditions hold.

The demand for each good is equal to its supply,

$$\pi_i \sum_{j, i \in S(j)} f_{i,j}(l_{i,j}) = \sum_j d_{i,j} w_j L_j = \sum_j d_{i,j} N_j.$$

Each active industry makes a profit of zero, or

$$\pi_j f_{i,j}(l_{i,j}) = w_j l_{i,j} \text{ for } i \in S(j).$$

During the last decade, many papers have been written containing existence theorems for economic models in which production exhibits increasing returns to scale (see, for example, ref. 1). This model is a special case of these more general models. But some of the conditions required by these existence theorems are not satisfied in this model, so we need the theorem that follows. We make two assumptions about the production functions  $f_{i,j}$ .

1. Aside from a possible initial interval in which  $f_{i,j}(l_{i,j})$  is zero, average productivity  $f_{i,j}(l_{i,j})/l_{i,j}$  is continuous and strictly increasing.

2. Each country in autarchy produces a positive quantity of all goods. More succinctly,  $f_{i,j}(d_{i,j}L_j) > 0$  for all  $i, j$ .

**THEOREM 1.** *Under these assumptions, there will be an average-cost pricing equilibrium for any pattern of specialization in which each country is the sole manufacturer of at least one good. In this equilibrium each industry assigned to each country will produce positive quantities of output, and the wage rates in all countries are positive.*

Different equilibria associated with different patterns of specialization are natural in the presence of economies of

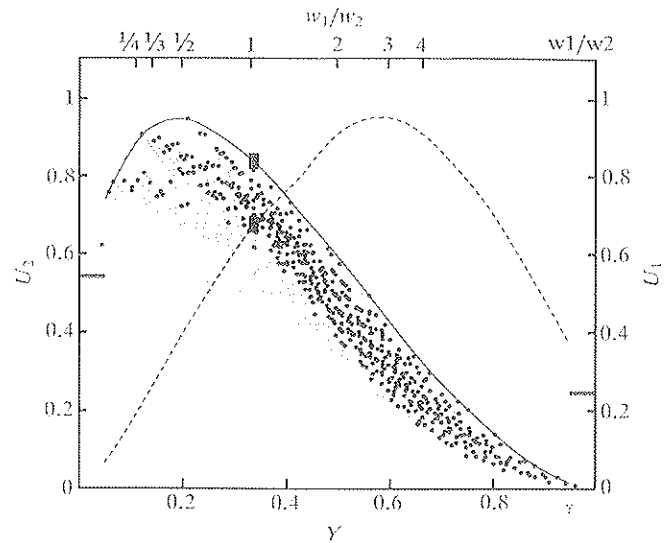


FIG. 2. Comparison of utilities of country 1 and country 2. ■, Ricardo points.

scale, since we are inevitably involved with many equilibria. Although the pattern of production at any one of these equilibria cannot be expected to be stable against large changes that move to the neighborhood of another equilibrium point, they can reasonably be expected to be stable against sufficiently small changes.

This motivates a further mild restriction on the production functions that ensures that nonproducers make a negative profit near equilibrium. More precisely assume that  $\lim_{l_{i,j} \rightarrow 0} f(l_{i,j})/l_{i,j} = 0$ . Then if  $l_{i,j} = 0$  at some equilibrium point with prices  $\pi_i$  and wages  $w_j$  we will have  $\pi_i f_{i,j}(l_{i,j}) - w_j l_{i,j} < 0$  for some interval  $0 < l_{i,j} < I(i, j)$ . In what follows, we will always assume this restriction on our production functions.

Clearly this condition is satisfied for all production functions of the form  $f(l) = e^{l^\alpha}$  with  $\alpha > 1$ , as well as by any  $f(l)$  that is zero for an interval to the right of the origin. It does not hold for the Ricardo case  $e^{l^\alpha}$  with  $\alpha = 1$ , but it does hold if that function is preceded by an interval of zero output.

**Two Countries: The Array of Solutions**

For two countries the existence theorem provides us with at least  $3^n - 2^{n+1} + 1$  equilibria,  $2^n - 2$  of which are locally stable in the sense just given. We now turn to the analysis of this array of possible outcomes.

The problem of finding the shape of the array of solutions in  $U-Y$  space is closely related to the problem of finding specialized (i.e., one producer) solutions that have high utility. To find these we introduce an integer programming formulation. Using  $N_j = w_j L_j$  for national income, the supply equals demand equation is

Table 1. Data for a typical two-country system

	Product								
	1	2	3	4	5	6	7	8	9
Demands									
Country 1	0.10	0.10	0.21	0.14	0.22	0.04	0.06	0.13	0.07
Country 2	0.05	0.21	0.11	0.15	0.23	0.07	0.08	0.10	0.20
Efficiencies									
Country 1	0.52	0.71	0.91	0.92	1.01	1.23	1.30	1.02	0.30
Country 2	1.00	1.02	0.70	0.94	1.24	0.60	0.70	0.77	0.50
Production exponent	1.30	1.50	1.70	1.90	2.00	2.00	2.10	2.00	1.61

Country 1, labor supply = "product" 8; country 2, labor supply = "product" 4. Production function =  $e_{i,j} l^{\alpha}$ . Demands shown are renormalized to total 1 in actual computation.

$$d_{i,1}N_1 + d_{i,2}N_2 = w_1l_{i,1} + w_2l_{i,2}. \quad [1]$$

We now introduce variables  $x_i$ ,  $0 < x_i < 1$ , that represent the fraction of the total value of demand for the  $i$ th product that is paid as wages in country 2. By definition then,

$$x_i(d_{i,1}N_1 + d_{i,2}N_2) = w_2l_{i,2}. \quad [1.1]$$

Summing the Eqs. 1 and using

$$\sum_i l_{i,j} = L_j$$

gives

$$\left(\sum_i d_{i,1}x_i\right)N_1 + \left(\sum_i d_{i,2}x_i\right)N_2 = N_2,$$

which we will refer to as the trade equation. This equation determines the wage ratio  $w_1/w_2$  once the  $x_i$  are known. Since it is only this ratio of wages that matters, it is often useful to deal with normalized variables  $Y = N_1/(N_1 + N_2)$ . Then,  $1 - Y$  is  $N_2/(N_1 + N_2)$  and  $0 < Y < 1$ . For fixed country sizes  $L_j$ , the ratio  $Y/(1 - Y)$  is proportional to the wage ratio. In terms of these normalized variables the trade equation becomes

$$\left(\sum_i d_{i,1}x_i\right)Y + \left(\sum_i d_{i,2}x_i\right)(1 - Y) = (1 - Y). \quad [2]$$

Once the  $x_i$  are given, the wages and national incomes are calculated from Eq. 2. The labor amounts are then calculated from Eq. 1, so supply equals demand, and the amount of each good produced is known. Because the trade equation is satisfied, the total labor supply in each country is used.

If one more condition is satisfied,  $x$  will be an equilibrium point. That condition is that all producers who produce at a positive level produce at equal average cost. While most  $x$  do not satisfy this condition, all integer (i.e., 0, 1)  $x$  do, since the entire wage bill is in one country and there is only one producer of each good. So integer  $x$  are equilibria. Of course there are equilibria with more than one producer of a single good as well and we will return to these later. However the integer equilibria do largely determine the shape of the solution array.

The utility for country 1 associated with the equilibrium point  $x$  can be written

$$u(x, Y) = \ln U(x, Y) = \sum_i d_{i,1} \ln F_i(Y) Q_i(x, Y)$$

with

$$F_i(Y) = \frac{d_{i,1}Y}{d_{i,1}Y + d_{i,2}(1 - Y)},$$

$$Q_i(x, Y) = q_{i,1}(1 - x_i, Y) + q_{i,2}(x_i, Y).$$

The  $F$  term gives the fraction of goods produced that is consumed in country 1, the  $Q$  term gives the quantity produced, which is the sum of the quantities produced in each country, and the  $q$  values are defined by  $q_{i,j}(x, Y) = f_{i,j}(l_{i,j})$  and by Eq. 1.1. When the  $x_i$  are integer all these expressions simplify considerably and the utility can be rewritten as

$$u(x, Y) = \sum_i d_{i,1} \ln F_i(Y) + \sum_i d_{i,1} \ln q_{i,1}(1, Y) + \sum_i x_i d_{i,1} \ln \frac{q_{i,2}(1, Y)}{q_{i,1}(1, Y)}.$$

For fixed  $Y$  this expression is linear in the  $x_i$ .

To find the upper boundary of the solution array we define the function  $B_j(Y)$  to be the result of maximizing  $u(x, Y)$  subject to the trade equation (or a variant of it) and subject to the  $x_i$  being 0 or 1. Finding this maximal  $u(x, Y)$ , for any given  $Y$ , is an integer programming problem, in fact a knapsack problem, and it will be described explicitly below. The  $B_j(Y)$  values obtained this way would certainly be equal to or above any integer equilibrium point for that  $Y$ .

This maximization problem also has economic meaning. Once  $Y$  is fixed, the demand in both countries for any good is fixed. Hence the fraction of the total production of any good that goes to each country is also known. For fixed  $Y$  the only way to improve the utility from any one good is to assign the production to the more-efficient producer to increase the quantity produced. The coefficients in the objective function of the knapsack problem then measure the improvements in utility that come from these assignments.

However there is only one  $Y$  for which it is possible to assign all the production to the more efficient producer; this is the Ricardo level mentioned above. For wage rates  $Y$  greater than the Ricardo level, were country 2 to be the sole producer of those goods it makes more efficiently at that wage rate, the demand for its labor would outstrip the supply. For  $Y$  below the Ricardo level, the same obtains for country 1. If we consider the derivation of the trade equation we see that a solution violating the labor constraint will also not satisfy the trade equation. So the problem of assigning production to the more-efficient producers, to maximize utility without violating the labor constraint, is what we have here in economic terms.

Precisely as written Eq. 2 will not usually have a solution for integer  $x$ . This reflects the fact that there are equilibria for certain wage ratios only. To deal with this difficulty we relax the trade equation to an inequality and define  $B_j(Y)$  by the integer programming problem,

$$B_j(Y) = \text{Max}_x u(x, Y) \quad x_i \text{ integer}, \quad 0 \leq x_i \leq 1$$

with

$$\sum_i \{d_{i,1}Y + d_{i,2}(1 - Y)\}x_i \leq (1 - Y).$$

In this trade inequality we have rearranged the terms to display the  $x_i$ . The inequality points as shown for  $Y$  above the Ricardo level and is reversed for  $Y$  below the Ricardo level. The relaxation allows the underutilization of labor in the country whose labor is scarce. Consequently the maximization of utility for the given  $Y$  should push the inequality very close to equality as the attempt is made to use this valuable labor.

The  $B_j(Y)$  so defined can be computed by ordinary dynamic programming techniques, and it allows the computation of the array boundary without examining the  $2^n$  specialized solutions. Furthermore the dynamic programming problem gives actual integer solutions, and hence equilibria, which can be expected to be close to the boundary curve.

In addition, there is an even easier calculation for getting a weaker boundary curve, which we will call  $B(Y)$ . To get  $B(Y)$  we further relax the problem by allowing continuous  $x_i$ . It is easily seen that with continuous variables the trade inequality will always be satisfied as an equality, so in fact  $B(Y)$  is given by the maximization of  $u(x, Y)$  subject to Eq. 2.

The solution technique is now particularly simple. The problem can be thought of as filling a space of length  $(1 - Y)$  with amounts  $x_i$  of goods each of length

$$d_{i,1}Y + d_{i,2}(1 - Y) \quad \text{with value} \quad d_{i,1} \ln \frac{q_{i,2}(1, Y)}{q_{i,1}(1, Y)}.$$

The solution to such a problem is to put in goods in succession in the order of their value per unit length, until the space is exactly used up. It is the results of these simple calculations that appear as the dotted lines in Figs. 1 and 2.

Although both calculations appear to be very effective in bounding the solution array in actual calculations, we will also make a more precise statement about solutions near the boundary curve. For production functions of the form

$$f(i,j) = e_{i,j} l^{\alpha_i} \quad \alpha_i > 1 \quad [3]$$

the following holds.

**THEOREM 2.** For any point on  $B(Y)$  with coordinates  $Y'$ ,  $B(Y')$ , there is a (nearby) integer solution  $x$  such that

$$|Y' - Y(x)| \leq \frac{\delta}{(1 - \rho)} |B(Y') - u(x, Y(x))| \leq \delta K(Y', \alpha, \beta, \rho),$$

where

$$\delta = \max_i d_{i,1} \quad \alpha = \max_i \alpha_i \quad \beta = \max_i \frac{L_2 e_{i,2}}{L_1 e_{i,1}}$$

$$\rho = \sum_i (d_{i,1} - d_{i,2}) \quad i \text{ such that } (d_{i,1} - d_{i,2}) > 0$$

and

$$K(Y', \alpha, \beta, \rho) = \alpha \ln \frac{Y(x)\beta}{1 - Y(x)} + \frac{\sum_i d_{i,1} \alpha_i}{1 - \rho} \left( \frac{1}{Y(x)} + \frac{1}{1 - Y'} \right).$$

Note that  $\rho$  can be 1 only for the special case of orthogonal demands; i.e.,  $d_{i,1} d_{i,2} = 0$  all  $i$ . The same result holds with  $B_i(Y')$  replacing  $B(Y')$ .

Note especially that any sequence of problems for which  $\alpha$ ,  $\beta$ , and  $1/(1 - \rho)$  are bounded, while  $\delta$  approaches zero, will have solutions approaching the various boundary curves at every point. In particular if the number of products increases in such a way that each one absorbs a decreasing fraction of national income, while the other parameters remain bounded, every point of both  $B_i(Y)$  and  $B(Y)$  will be approached by equilibrium points.

**Other Equilibria**

So far we have worked entirely with integer solutions, that is to say with specialized solutions. Some justification for this approach can be seen from the following theorem.

**THEOREM 3.** Let  $x$  be any equilibrium solution, whether specialized or not. Let  $Y(x)$  be the corresponding  $Y$ . If  $u(x, Y)$  is the utility of  $x$  to country 1, then

$$u(x, Y) \leq B_1(Y) \leq B(Y).$$

So all the equilibrium points, not just the specialized ones, lie under the boundary curves.

Nonspecialized solutions are harder to analyze than are the specialized. Fortunately they are connected by *Theorem 3*. In addition, empirical work, as illustrated in Figs. 1 and 2, shows the generally lower utility of nonspecialized solutions.

However, as the existence proof shows, this model does allow for mixed equilibria, they exist and they are numerous. The simplest case of a mixed equilibrium is the one in which only one good, say  $x_1$ , is produced in both countries. Let

$x(x_1) = (x_1, x')$ , where  $x'$  is a fixed integer  $m - 1$  vector. For this case, and with the restriction to production functions of form 3, the following theorem can be stated.

**THEOREM 4.** If  $e_{1,1}/w_{1,1} < e_{1,2}/w_{1,2}$  for both  $x(0)$  and  $x(1)$ , or if  $e_{1,1}/w_{1,1} > e_{1,2}/w_{1,2}$  for both  $x(0)$  and  $x(1)$ , then there is only one intermediate solution  $x(x_1)$ ,  $0 < x_1 < 1$ .

That this condition cannot be wholly dispensed with is shown by the following partial converse:

**THEOREM 5.** If the condition of the uniqueness theorem is not met, there are always multiple equilibria for values of the production exponent  $\alpha_i$  sufficiently close to 1.

These intermediate equilibrium points have their own properties of local stability, and they can be characterized in a plausible way as stable and unstable. When there are multiple intermediate equilibria, it can be shown that there are always an odd number of them and that some are stable.

**Observations About the Model**

Let us substitute for a production function  $f_{i,j}$  a new production function  $f'_{i,j}$  with  $f_{i,j}(l) = f'_{i,j}(l)$  for  $l$  greater than the autarchy labor level  $d_{i,j} L_j$ . We assert (without proof) that the array of specialized solutions and their boundary curves are unchanged. What has changed is the stability of the specialized solutions. If we were to use a  $f'$  with sharp positive slope at the 0 level, the function would no longer satisfy the limit condition assumed above. Or if we were to use a  $f'$  that is zero till near the autarchy labor level, and then rises rapidly to its former shape, we would have built in a strong barrier to entry.

If we substitute for country 2 a different country 2 with the same demand function, and the same production functions, but with a larger labor force, we assert that all specialized solutions for country 1, with the exception of autarchy, will improve in utility. This is illustrated in Fig. 2. The larger country does less well relative to autarchy than its smaller trading partner.

Within this model we can look at specialized solutions that differ in one industry only; i.e., country 1 makes product  $j$  in one solution, country 2 makes that product in a second solution, and all other products are made as before. In the notation of the section above we can compare the national incomes and utilities of  $x(0)$  and  $x(1)$ . The effect on national income is to increase it in country 2 and decrease it in country

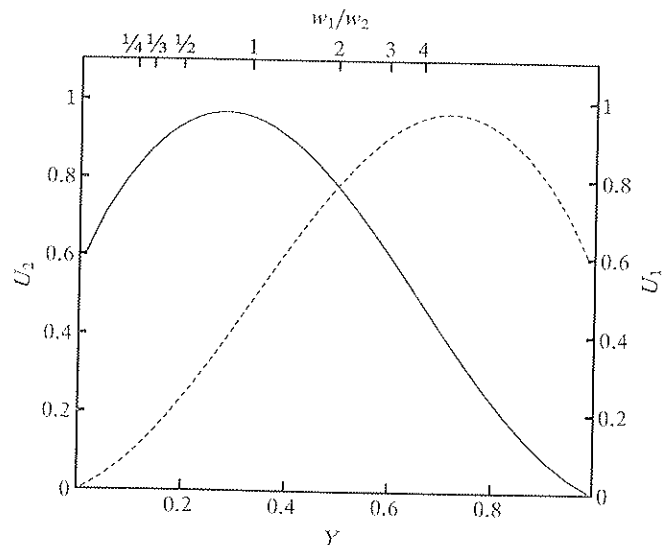


FIG. 3. Utility of country 1 and country 2 for the special case of symmetrical demands and production function  $f_{i,j}(l) = e_{i,j} l$  for  $\alpha = 1.5$ . —, Country 1; ---, country 2.

1. However utility can either increase or decrease and no simple general statement can be made.

### Special Case

While there are several interesting special cases, we will confine ourselves here to discussing one that sheds some light on the characteristic shape of the boundary curve. We consider the case of symmetrical demands and production function  $f_{i,j}(l) = e_{i,j} l^\alpha$  for a fixed exponent  $\alpha$ . After some reasoning the function to be maximized simplifies to

$$u(x, Y) = K + (\alpha - 1) \ln \frac{1}{Y} + \alpha(1 - Y) \ln \frac{Y}{(1 - Y)} \\ + \text{Max}_x \sum_i x_i d_{i,1} \ln \frac{L_2^\alpha e_{i,2}}{L_1^\alpha e_{i,1}},$$

with

$$K = \sum_i d_{i,1} \ln d_{i,1} L_2^\alpha e_{i,2},$$

which is the utility of country 1 in autarchy.

In this expression the part involving  $x$  now has coefficients independent of  $Y$ . As a consequence we can solve this once

for all  $Y$ , either in the dynamic programming case to get  $B_f(Y)$  or in the continuous knapsack case to get  $B(Y)$ . It is also true that the integer solutions that are obtained for various  $Y$  in the course of the calculations all lie directly on the respective boundary curves.

Since we are solving a knapsack problem of length  $(1 - Y)$  we can express the result as a density  $d(Y)$  times the length.  $d(Y)$  will increase monotonically and be bounded above and below by constants representing the greatest and least possible densities. Upon rearranging terms we get for the utility,

$$U(Y) = e^K Y \left\{ \left( \frac{1}{Y} \right)^Y \left( \frac{1}{(1 - Y)} \right)^{(1 - Y)} \right\}^\alpha \exp(1 - Y) d(Y). \quad [4]$$

Competition among identical countries makes perfect sense in our model. In that case the  $d(Y)$  term will be 0. The resulting curve

$$U(Y) = KY \left\{ \left( \frac{1}{Y} \right)^Y \left( \frac{1}{(1 - Y)} \right)^{(1 - Y)} \right\}^\alpha$$

is plotted in Fig. 3 for  $\alpha = 1.5$ . It exhibits the characteristic boundary shape, one from which Eq. 4 can deviate only slightly.

1. (1988) *J. Math. Econ.* 17, nos. 2 and 3.