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The atoms of integer programming

Ralph E. Gomory

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1 Introduction

After thirty years of doing other things, I am once again doing some research on integer programming. It has been interesting and exciting for me to see what has changed during those thirty years.

The practical side of integer programming has developed far more than I would have ever anticipated. Problems of a size and complexity that we would never have thought of attempting thirty years ago are being done routinely. And this has been accomplished not by some theoretical or conceptual breakthrough (the branch and cut methods generally used are fundamentally the same as before) but by the intelligent and persistent use of empirical methods and empirical learning. Added to this, of course, is 30 years of progress in electronics, which is worth another improvement factor of roughly 1000 in performance.

On the theoretical side there is continued interest in special problems: the traveling salesman problem, for example, is very much alive and well, as are many graph theory problems. There is a large gap between the work on special problems and the work on large practical problems. The type of discourse in these two areas is quite different. One area deals in large test problems and how they run, the other is much more theoretical.

2 Theory and experiment

During my long tenure as a Director of Research of IBM, I had many years of close exposure to physical scientists. I came to appreciate the synergy of experiment and theory, which is sometimes possible in physics. This is very different from the sort of experiment many of us did in our high school or early college years. In high school or college we went into the physics lab to verify some law of nature already described in our textbooks. But during my

R. E. Gomory (✉)
Alfred P. Sloan Foundation, 639 Fifth Avenue, New York, NY 19111, USA
e-mail: gomory@sloan.org

extended exposure to physics and physicists I saw something quite different from that. I saw experimental results that suggested theory, often a rough theory, that shed partial light on the current experiment. The rough theory suggested other experiments yet to be done, and those in turn led to changes in the theory. This sort of step by step progress led to the discovery of high-temperature superconductivity in 1986. A rough theory suggested trying for superconductivity in certain classes of materials, eventually experiment turned up something that worked a bit, more experiment turned up better stuff, theory today can still not explain in an acceptable way why we have this effect at all. Experiment is still ahead of theory. But through their interaction really important progress was made.

With computers, this sort of experimental work is now possible in some areas of mathematics. But in our area the two approaches, the experimental and the theoretical usually seem quite far apart. However there is one place where I think they could come together—this is of course an area in which I have been interested for a long time.

3 Corner polyhedra

I am referring to thr possibility of both theoretical and experimental work on Corner Polyhedra. Corner polyhedra, introduced in Gomory (1969), are obtained as a relaxation of the Integer Programming (IP) problem. Specifically at a vertex V of a Linear Programming (LP) problem, the constraint that the basic variables be non-negative is dropped. The geometric object so obtained is the Corner Polyhedron. This is the larger of the two filled areas in Figure 1.

Corner Polyhedra tend to be closely related to their integer programs. This is not surprising since, as the figure suggests, Corner Polyhedra tend to be identical with the original (not relaxed) Integer Polyhedra of an integer LP problem for some[-21pc] distance around the

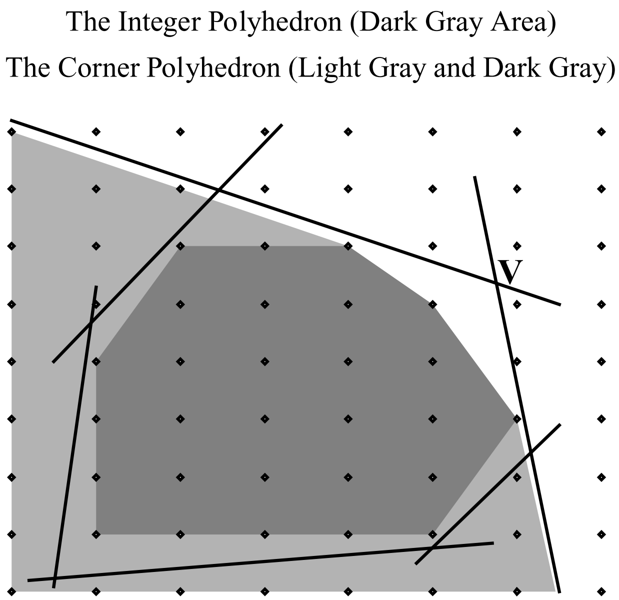

Fig. 1

vertex. So Corner Polyhedra have many of the characteristics of the polyhedron that is defined by the integer programming problem itself. Corner Polyhedra become identical with the IP problem for sufficiently large right hand sides b. There are even classes of integer programs where maximizing over the Corner Polyhedron solves the original IP for all right hand sides, as Serkan Hosten and Rheka R. Thomas have shown in Hosten and Thomas (2003).

Unlike general integer programs, Corner Polyhedra come in a natural sequence ranging from very small and very simple to Corner Polyhedra that are arbitrarily large and complex. Each Corner Polyhedron is associated with a group, and as the groups get bigger the Corner Polyhedron becomes more complex.

Work in this area is susceptible to (computational) experiment. With today's easy to use computers, experimenting is easy. Experiment shows that Corner Polyhedra, which are present in every integer program, have lots of structure. They are closely linked to finite groups and their facets can be discovered and tabulated, remarkably enough, by ordinary linear programming. Shortest path methods can also be used. Since these polyhedra are present in every integer program, information about them relates to general integer programming. One very direct connection is that the facets of the Corner Polyhedra provide cutting planes for general integer programming. The Gomory mixed integer cut turns out to be the simplest of these facets but there are many many more.

Cutting planes that I could never have imagined come popping out of calculations made possible by the theory of Corner Polyhedra and the subsequent work in the area by Ellis Johnson and myself. Gomory and Johnson (2003) contains an account of some experimental work as well as many other references to Corner Polyhedra. In the area of Corner Polyhedra there are wide-open theoretical questions, and as we go to larger and larger Corner Polyhedra, our "shooting experiments" help us to see where the complexity and difficulty of IP really comes from.

4 Corner polyhedra and groups

What do Corner Polyhedra have to do with groups? Despite the geometry shown in the figure, the natural definition of a Corner Polyhedron has nothing to do with either LP or IP. The most straightforward definition is as follows: In a group G (which we will discuss below) with group elements g, we define a *path* to some goal element g_0 in G as a finite sum of group elements adding up to the selected goal element g_0. Let $t(g)$ be the number of times group element g is used in a path. If we form a space, which we will call *path space,* with one dimension in path space for each group element g, the vectors $\{t(g)\}$ that form paths to g_0 are among the integer lattice points of that space. Each integer lattice point represents a path to *some* group element. We select from among these lattice points the lattice points p that form paths to g_0.

The convex hull of the points p turns out to be what we have described geometrically as the (Master) Corner Polyhedron. What we call here path space is what we call T-Space in the geometrical theory. The connection with Integer Programming comes from the fact that we can map, in a natural way, solutions to the IP problem into paths in the group G.

I have found working with Corner Polyhedra in today's environment an exciting mixture of theory and revealing computational experiment. Doing this work reminds me that when I discovered Corner Polyhedra long ago, I thought to myself, "these are the atoms of integer programming." Looking at the experiments today, I still think so.

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